Branch Flow Model for Radial Networks: Convex Relaxation

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***Abstract*—Power flow optimization is generally nonlinear and non-convex, and a second-order cone relaxation has been proposed recently for convexification. We prove several sufficient conditions under which the relaxation is exact. One of these conditions seems particularly realistic and suggests guidelines on integrating distributed generations.**

I. Introduction

The bus injection model is the standard model for power flow analysis and optimization. It focuses on nodal variables such as voltages, current and power injections and does not directly deal with power flows on individual branches. A key advantage is the simple linear relationship $I = YV$ between current injections I and bus voltages V through the admittance matrix Y. Instead of nodal variables, branch flow models focus on currents and powers on the branches. It has been used mainly for modeling distribution circuits which tend to be radial, but has received far less attention. In [1], we advocate the use of a branch flow model for both radial and mesh networks, and demonstrate how it can be used for optimizing design and operation of power systems, including optimal power flow, demand response, and Volt/VAR control.

The optimal power flow (OPF) problem seeks to minimize a certain cost function, such as power loss and generation cost, subject to physical constraints including Kirchoff's laws, heat constraints, as well as voltage regulation constraints. There has been a great deal of research on OPF since Carpentier's first formulation in 1962 [2]; surveys can be found in, e.g., [3]–[7]. OPF is generally nonconvex and NP hard, and a large number of optimization algorithms and relaxations have been proposed. A popular approximation is the DC power flow, which is a linear program and therefore easy to solve, e.g. [8]–[11]. An important observation was made in [12], [13] that the full AC OPF can be formulated as a quadratically constrained quadratic program and therefore can be approximated by a semidefinite program. While this approach is illustrated in [12], [13] on several IEEE test systems using an interior-point method, whether or when the semidefinite relaxation (SDR) will turn out to be exact is not studied. A sufficient condition is derived in [14] under which the SDR is exact. Moreover this condition is shown to essentially hold in various IEEE test systems. This result is further extended in [15] to include other variables and constraints and in [16] to exploit the sparsity of power networks. While this line of research has generated a lot of interest, limitations of the SDR have recently been studied in [17] using 3, 5, and 7-bus examples. They show that as a line-flow constraint is tightened, the sufficient condition in [14] fails to hold for these examples and the relaxation gap becomes nonzero. Moreover, the solutions produced by the SDR are physically meaningless in those cases. Indeed, examples of nonconvexity have long been discussed in the literature, e.g., [18]–[20]. Remarkably, it turns out that if the network is radial, then the sufficient condition of [14] always holds, provided that the bounds on the power flows satisfy a simple pattern, as proved independently in [21], [22]. This is important as almost all distribution systems are radial networks.

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Indeed, for radial networks, different convex relaxations have also been studied using branch flow models. The model considered in this paper is first proposed in [23], [24] for the optimal placement and sizing of switched capacitors in distribution circuits for Volt/VAR control. Recasting their model as a set of linear constraints together with a set of quadratic equality constraints, [25] proposes a second-order-cone (SOC) convex relaxation, and proves that the relaxation is exact for radial networks, when there are no upper bounds on the loads. See also [26] for an SOC relaxation of a linear approximation of the model in [23], [24], and [27]–[29] for other branch flow models.

Removing upper bounds on the load may be unrealistic, e.g., in the context of demand response. In this paper, we prove that the SOC relaxation is exact for radial networks, provided there are no upper bounds on the voltage magnitudes and one of the following conditions holds:

(1) Both real power and reactive power flow unidirectionally from the substation to the branches. This condition holds when there is no or little distributed generation in the circuit, and no shunt capacitor is turned on.
(2) The real power flows unidirectionally from the substation to the branches, and the resistance to reactance ratio of the distribution line is non-decreasing as it branches out from the substation. The condition on the resistance to reactance ratio is satisfied in most distribution circuits. Hence, this condition holds even when some shunt capacitors are turned on, as long as distributed generation in the circuit remains small.
(3) The reactive power flows unidirectionally from the substation to the branches, and the resistance to reactance ratio is non-increasing as the line branches out.
(4) The resistance to reactance ratio is uniform throughout the distribution circuit.

Condition (2) seems realistic and confirms the popular guideline in practice on integrating distributed generations: they should be spaced on the circuit in a way that avoids reverse real power flow. This ensures exact relaxation and greatly simplifies optimal distribution circuit control.

In this paper we first present the branch flow model in section II, and then prove a variety of conditions under which the SOC relaxation of the model is exact, for radial networks when there are no upper bounds on bus voltages, in section III. Finally, we use a real-world distribution circuit to illustrate our sufficient conditions in section IV.

II. Problem formulation

A. Branch flow model

Consider a radial distribution circuit that consists of a set N of buses and a set E of distribution lines connecting these buses. We index the buses in N by $i = 0, 1, \ldots, n$, and denote a line in E by the pair (i, j) of buses it connects. For convenience, we put the bus i that is closer to the substation in front. Let bus 0 represents the substation, which has fixed voltage and flexible real and reactive power injection to balance the electricity demand; other buses in N represent branch buses, whose voltage varies according to load conditions.

For each line $(i, j) \in E$, let I_{ij} be the complex current flowing from buses i to j, $z_{ij} = r_{ij} + \mathbf{i}x_{ij}$ be the impedance on line (i, j), and $S_{ij} = P_{ij} + \mathbf{i}Q_{ij}$ be the complex power flowing out from buses i to bus j. For each bus $i \in N$, let V_i be the complex voltage on bus i and s_i be the complex net load—consumption minus generation—on bus i.[1] For power flow analysis, $s := (s_1, \ldots, s_n)$ is a given constant. For power flow optimization (such as Volt-VAR control and demand response), s is a control variable.

TABLE I: Notations.

V_i, v_i	complex voltage on bus i with $v_i = \lvert V_i\rvert^2$
$s_i = p_i + \mathbf{i}q_i$	complex net load on bus i
I_{ij}, ℓ_{ij}	complex current from buses i to j with $\ell_{ij} = \lvert I_{ij}\rvert^2$
$S_{ij} = P_{ij} + \mathbf{i}Q_{ij}$	complex power flowing out from buses i to bus j
$z_{ij} = r_{ij} + \mathbf{i}x_{ij}$	impedance on line (i, j)
a^*	complex conjugate of a

Power flow at steady states satisfies:

- power balance at each bus $j \in N\backslash\{0\}$ with $(i, j) \in E$:

$$\left(S_{ij} - z_{ij}|I_{ij}|^2\right) - \sum_{k:(j,k)\in E} S_{jk} \;=\; s_j; \qquad (1)$$

- Ohm's law on each line $(i, j) \in E$:

$$V_i - V_j \;=\; z_{ij} I_{ij}; \qquad (2)$$

- definition of complex power on each line $(i, j) \in E$:

$$S_{ij} \;=\; V_i I_{ij}^*. \qquad (3)$$

We refer to (1)–(3) as the *branch flow model*. As customary, we assume that the complex voltage V_0 on the substation bus is given. Recall that $|N| = n + 1$ and define $m := |E|$. The branch flow equations (1)–(3) specify $2m + n$ nonlinear equations in $2m + n$ complex variables $(S, I, V) := (S_{ij}, I_{ij}, (i, j) \in E,\ V_j, j = 1, \ldots, n)$, when $s_1, \ldots, s_n$ are specified. The solutions of (1)–(3) define the steady states of the power system.

B. Relaxed branch flow model

For notational simplicity, define $\ell_{ij} := |I_{ij}|^2$ and $v_i := |V_i|^2$. Then (2) and (3) imply $V_j = V_i - z_{ij}S_{ij}^*/V_i^*$. Taking the magnitude squared, we have $v_j = v_i + |z_{ij}|^2\ell_{ij} - (z_{ij}S_{ij}^* + z_{ij}^*S_{ij})$. Using (1) and (3) and in terms of real variables, we therefore have

$$p_j \;=\; P_{ij} - r_{ij}\ell_{ij} - \sum_{k:(j,k)\in E} P_{jk},\ j = 1, \ldots, n \qquad (4)$$

$$q_j \;=\; Q_{ij} - x_{ij}\ell_{ij} - \sum_{k:(j,k)\in E} Q_{jk},\ j = 1, \ldots, n \qquad (5)$$

$$v_j \;=\; v_i - 2(r_{ij}P_{ij} + x_{ij}Q_{ij}) + (r_{ij}^2 + x_{ij}^2)\ell_{ij}, \quad (i, j) \in E \qquad (6)$$

$$\ell_{ij} \;=\; \frac{P_{ij}^2 + Q_{ij}^2}{v_i}, \qquad (i, j) \in E. \qquad (7)$$

These equations (4)–(7) were first proposed in [23], [24] to model radial distribution circuits. We will refer to them as the *relaxed branch flow model*. They define a system of equations in the variables $(P, Q, \ell, v) := (P_{ij}, Q_{ij}, \ell_{ij}, (i, j) \in E,\ v_i, i = 1, \ldots, n)$, which are a subset of the original (complex) variables (S, I, V), without the angles $\angle V_i, \angle I_{ij}$. In contrast to the original branch flow equations (1)–(3), the relaxed system (4)–(7) specifies $2(m+n)$ equations in $3m+n$ real variables (P, Q, ℓ, v), for a given $(p_i, q_i, i = 1, \ldots, n)$. For a radial network, $m = |E| = |N| - 1 = n$. Hence the relaxed system (4)–(7) specifies $4n$ equations in $4n$ real variables, given a $(p_i, q_i, i = 1, \ldots, n)$.

One may consider (P, Q, ℓ, v) as a projection of (S, I, V) where each variable I_{ij} or V_i is relaxed from a point in the complex plane to a circle with the same radius. Then the question is that given a solution (P, Q, ℓ, v) of the relaxed model (4)–(7) whether one can always recover a solution (S, I, V) of the original branch flow model (1)–(3). In [1], it is proved that this is indeed possible when the network is radial and two simple algorithms are provided to compute the unique angles $\angle V_i, \angle I_{ij}$ together with which (P, Q, ℓ, v) solves (1)–(3). Indeed there is one-one correspondence between the solution set of (1)–(3) and its relaxed equations (4)–(7) when the network is radial. However this is not the case for mesh networks [1].

Hence, for radial networks, the key to power flow analysis and optimization is the relaxed model (4)–(7).

C. Optimal power flow

Consider the problem of minimizing power loss $\sum_{(i,j)\in E} r_{ij}\ell_{ij}$ over the network where the optimization variables are $p := (p_1, \ldots, p_n)$, $q := (q_1, \ldots, q_n)$ as well as (S, I, V). p_i and q_i are the net real and reactive power consumption at node i. They can be either negative or positive depending on whether the node represents a generator, a load, a distributed energy resource, or a shunt capacitor. For instance, [23], [24] formulates a Volt-VAR control problem for a distribution circuit where q_i represents the placement and sizing of shunt capacitors; [25] uses it for inverter-based Volt-VAR control and formulates it as optimization over reactive power generations q_i from inverters that depend on the solar power output p_i at nodes i.

Since the objective function does not involve the angles $\angle V_i, \angle I_{ij}$ and, as discussed above, there is an one-one correspondence between the solutions of the branch flow model and its relaxed model, we can equivalently minimize power loss over the bigger feasible set defined by (4)–(7) instead of (1)–(3).

[1] Notice that s_0 denotes the complex power injected at the substation to balance the electricity demand in distribution circuit.

In addition to power flow equations (4)–(7), we impose the following constraints on power consumption or generation:

$$\underline{p}_i \le p_i \le \overline{p}_i, \quad \underline{q}_i \le q_i \le \overline{q}_i, \qquad i = 1, \ldots, n. \tag{8}$$

In particular, any of p_i, q_i can be a fixed constant by specifying that its upper and lower bounds coincide. Finally the voltage magnitudes must be maintained to be above a threshold:

$$\underline{v}_i \quad \le \quad v_i, \qquad i = 1, \ldots, n. \tag{9}$$

Given voltage v_0, the loss minimization problem is then:
LMP:

$$\begin{aligned} \text{min} \quad & \sum_{(i,j)\in E} r_{ij}\ell_{ij} \\ \text{over} \quad & (P, Q, \ell, v, p, q) \\ \text{s.t.} \quad & (4) - (7),\ (8) - (9). \end{aligned}$$

In general, finding the power flow solution that minimizes the power loss is NP hard because of the quadratic equality constraint (7).

III. Second-order cone relaxation

Following [25], we consider the following convex relaxation of LMP.
RLMP:

$$\begin{aligned} \text{min} \quad & \sum_{(i,j)\in E} r_{ij}\ell_{ij} \\ \text{over} \quad & (P, Q, \ell, v, p, q) \\ \text{s.t.} \quad & (4) - (6),\ (8) - (9) \\ & \ell_{ij} \ge \frac{P_{ij}^2 + Q_{ij}^2}{v_i}, \quad (i,j) \in E. \end{aligned} \tag{10}$$

By relaxing the equality constraints (7) in LMP to the inequality constraints (10), RLMP is a second-order cone program which is convex and can be solved efficiently. Clearly RLMP provides a lower bound on LMP. [25] proves that the relaxation is exact when there are no upper bounds on the real and reactive power consumptions in (8), but allows upper bounds on the voltages in (9). Our main result provides a variety of conditions that guarantee exact relaxation.

Let LMP(p,q) and RLMP(p,q) denote the optimization problems corresponding to LMP and RLMP, that fix (p,q) and optimize over the rest of the variables (ℓ, P, Q, v). If for every feasible (p,q), LMP(p,q) and RLMP(p,q) are equivalent, i.e., a point (ℓ, P, Q, v) is optimal for RLMP(p,q) if and only if it is optimal for LMP(p,q), then LMP and RLMP are equivalent. Hence, we are mainly interested in exploring the equivalence of LMP(p,q) and RLMP(p,q). If we ignore the terms associated with ℓ_{ij}, then (4)–(5) reduce to

$$P_{ij} \;=\; p_j + \sum_{k:(j,k)\in E} P_{jk} \tag{11}$$

$$Q_{ij} \;=\; q_j + \sum_{k:(j,k)\in E} Q_{jk} \tag{12}$$

for every $(i,j) \in E$. Note that equations (11)–(12) can be solved efficiently without solving RLMP(p,q), and we denote its solution by $(P^{nom}(p,q), Q^{nom}(p,q))$. Actually, P^{nom} only depends on p and Q^{nom} only depends on q.

Theorem 1: *Problems LMP(p,q) and RLMP(p,q) are equivalent, provided any of the following conditions hold:*

(1) $P_{ij}^{nom}(p) \ge 0,\ Q_{ij}^{nom}(q) \ge 0$ *for each* $(i,j) \in E$;
(2) $P_{ij}^{nom}(p) \ge 0$ *for each* $(i,j) \in E$; *and* $\frac{r_{ij}}{x_{ij}} \le \frac{r_{jk}}{x_{jk}}$ *for each triplet* (i,j,k) *such that* $(i,j) \in E$ *and* $(j,k) \in E$;
(3) $Q_{ij}^{nom}(q) \ge 0$ *for each* $(i,j) \in E$; *and* $\frac{r_{ij}}{x_{ij}} \ge \frac{r_{jk}}{x_{jk}}$ *for each triplet* (i,j,k) *such that* $(i,j) \in E$ *and* $(j,k) \in E$;
(4) $\frac{x_{ij}}{r_{ij}} = \frac{x_{hk}}{r_{hk}}$ *for each* $(i,j) \in E$ *and each* $(h,k) \in E$.

We interprete those conditions in Theorem 1 as follows. Firstly notice that the values $P^{nom}(p)$ and $Q^{nom}(q)$ are the real and reactive power flows without considering the losses along the line, and can be computed easily using (11)–(12). Hence, all four conditions in Theorem 1 can be easily checked without solving the relaxed problem RLMP(p,q).

Condition (1) roughly says that both the real and reactive power flow unidirectionally from the substation to the branches, i.e., there is no reverse flow. If there is no distributed generation (such as solar panels on rooftops), or if distributed generation is smaller than the downstream loads, then there is no reverse real power flow on all lines, i.e., $P_{ij}^{nom} \ge 0$ for all $(i,j) \in E$. If there are no shunt capacitors injecting reactive power, then distribution circuit consumes reactive power, and $Q_{ij}^{nom} \ge 0$ for all $(i,j) \in E$ consequently. However shunt capacitors are often turned on in peak load hours to regulate the voltage. In this case, reactive power flow can be negative on some of the lines. Therefore Condition (1) probably does not hold.

Condition (2) relaxes the restriction that $P_{ij}^{nom} \ge 0$, but imposes a new restriction that the ratio r/x must be nondecreasing along the distribution line. This condition on the impedance of distribution circuits is generally satisfied since the distribution line usually becomes thinner and thinner as it branches out from the substation. Hence, Condition (2) holds widely for radial distribution network as long as there are no distributed generations in the branches. Condition (2) is especially useful in Volt-VAR control, where real power consumption p is indeed a given constant. Even if there are distributed generations, Condition (2) still holds if those generations are smaller than their downstream loads, which is true for most distribution circuits.

When there are significant large distributed generations, causing reverse real power flow towards the substation on some of the lines, Condition (2) may not apply. Condition (4) further relaxes the constraints that $P^{nom} \ge 0$, but adds the new restriction that the ratio r/x be the same throughout the circuit. Hence, if the lines are uniform, then problem LMP and problem RLMP are equivalent, no matter how power flows on the lines.

After studying the gap between $LMP(p,q)$ and $RLMP(p,q)$ for each (p,q), we are ready to state the following corollary for the gap between LMP and $RLMP$.

Corollary 1: *Problems LMP and RLMP are equivalent, provided any of the following conditions hold:*

(1) $P_{ij}^{nom}(\underline{p}) \ge 0,\ Q_{ij}^{nom}(\underline{q}) \ge 0$ *for each* $(i,j) \in E$;
(2) $P_{ij}^{nom}(\underline{p}) \ge 0$ *for each* $(i,j) \in E$; *and* $\frac{r_{ij}}{x_{ij}} \le \frac{r_{jk}}{x_{jk}}$ *for each triplet* (i,j,k) *such that* $(i,j) \in E$ *and* $(j,k) \in E$;
(3) $Q_{ij}^{nom}(\underline{q}) \ge 0$ *for each* $(i,j) \in E$; *and* $\frac{r_{ij}}{x_{ij}} \ge \frac{r_{jk}}{x_{jk}}$ *for each triplet* (i,j,k) *such that* $(i,j) \in E$ *and* $(j,k) \in E$;
(4) $\frac{x_{ij}}{r_{ij}} = \frac{x_{hk}}{r_{hk}}$ *for each* $(i,j) \in E$ *and each* $(h,k) \in E$.

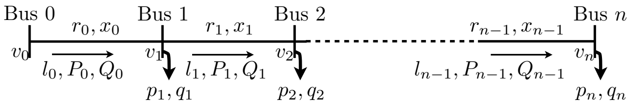

Fig. 1: A one-line distribution network.

For simplicity, we will demonstrate the proof of Theorem 1 for the case of a one-line distribution circuit (main feeder). The proof can be easily extended to a tree network. For a one-line network, we can abbreviate r_{ij}, x_{ij}, P_{ij}, Q_{ij}, l_{ij} by r_i, x_i, P_i, Q_i and l_i respectively as shown in without introducing ambiguity. Figure 1 describes the notations. We rewrite problems LMP(p,q) and RLMP(p,q) with the simplified notations as:

LMP-1:

$$\begin{aligned} \min \quad & \sum_{i=0}^{n-1} r_i \ell_i \\ \text{over} \quad & (P, Q, \ell, v) \\ \text{s.t.} \quad & P_i = r_i \ell_i + p_{i+1} + P_{i+1}, \\ & \qquad i = 0, \ldots, n-1 \qquad (13) \\ & Q_i = x_i \ell_i + q_{i+1} + Q_{i+1}, \\ & \qquad i = 0, \ldots, n-1 \qquad (14) \\ & v_{i+1} = v_i - 2(r_i P_i + x_i Q_i) + (r_i^2 + x_i^2)\ell_i, \\ & \qquad i = 0, \ldots, n-1 \qquad (15) \\ & \ell_i = \frac{P_i^2 + Q_i^2}{v_i}, \qquad i = 0, \ldots, n-1 \qquad (16) \\ & v_i \ge \underline{v}_i, \qquad i = 1, \ldots, n, \qquad (17) \end{aligned}$$

where $P_N := 0,\ Q_N := 0$, and

RLMP-1:

$$\begin{aligned} \min \quad & \sum_{i=0}^{n-1} r_i \ell_i \\ \text{over} \quad & (P, Q, \ell, v) \\ \text{s.t.} \quad & (13) - (15),\ (17) \\ & \ell_i \ge \frac{P_i^2 + Q_i^2}{v_i}, i = 0, \ldots, n-1. \end{aligned}$$

Note that p and q are treated as given constants hereafter. P^{nom} and Q^{nom} can be calculated as $P_i^{nom} = \sum_{j=i+1}^{n} p_j$ and $Q_i^{nom} = \sum_{j=i+1}^{n} Q_j$ for all $i = 0, \ldots, n-1$.

Lemma 1: *Problems LMP-1 and RLMP-1 are equivalent, provided any of the following conditions hold:*

1) $P_i^{nom} \ge 0,\ Q_i^{nom} \ge 0$ *for each* $i = 0, \ldots, n-1$*;*
2) $P_i^{nom} \ge 0$ *for each* $i = 0, \ldots, n-1$*; and* $\frac{r_i}{x_i} \le \frac{r_{i+1}}{x_{i+1}}$ *for* $i = 0, \ldots, n-2$*;*
3) $Q_i^{nom} \ge 0$ *for each* $i = 0, \ldots, n-1$*; and* $\frac{r_i}{x_i} \ge \frac{r_{i+1}}{x_{i+1}}$ *for* $i = 0, \ldots, n-2$*;*
4) $\frac{x_i}{r_i} = \frac{x_j}{r_j}$ *for each* $0 \le i, j \le n-1$.

Proof: Use (13)–(15) to write $P(\ell), Q(\ell)$ and $v(\ell)$ as functions of ℓ, we can rewrite RLMP-1 as

$$\begin{aligned} \min_{\ell} \quad & \sum_{i=0}^{n-1} r_i \ell_i \\ \text{s.t.} \quad & \ell_i \ge \frac{P_i(\ell)^2 + Q_i(\ell)^2}{v_i(\ell)},\ i = 0, \ldots, n-1 \qquad (18) \\ & v_i(\ell) \ge \underline{v}_i,\ i = 1, \ldots, n. \qquad (19) \end{aligned}$$

Associate Lagrangian multipliers $\lambda_i \ge 0$ with (18), $u_i \ge 0$ with (19). Then the Lagrangian of RLMP-1 is

$$\begin{aligned} L(l, \lambda, u) \quad = \quad & \sum_{i=0}^{n-1} r_i \ell_i + \sum_{i=0}^{n-1} \lambda_i \left(\frac{P_i^2 + Q_i^2}{v_i} - \ell_i \right) \\ & + \sum_{i=1}^{n} u_i \left(\underline{v}_i - v_i \right) \end{aligned}$$

If RLMP-1 is infeasible, neither is LMP-1. If RLMP-1 is feasible, then optimal ℓ^* exists according to Lemma 2 in the appendix. Hence, there exists dual variable $(\lambda^*, u^*) \ge 0$ such that (ℓ^*, λ^*, u^*) are primal dual optimal for problem RLMP-1 and satisfy the KKT condition. If any of the four conditions in Lemma 1 holds, then $\lambda^* > 0$ according to Lemmas 3-6 in the appendix. It follows from the complementary slackness that equality in (18) is attained, and ℓ^* is feasible for problem LMP-1. Then, ℓ^* is optimal for LMP-1 since ℓ^* solves the relaxed problem RLMP-1. Hence, a point (P^*, Q^*, ℓ^*, v^*) is optimal for RLMP-1 if and only if it is optimal for LMP-1.□

IV. Case Study

How generally the conditions provided in Theorem 1—to guarantee the equivalence between Problem LMP and RLMP—hold in pratical distribution circuits has been discussed after Theorem 1. Basically, Condition (1) holds if distributed generation is smaller than its downstream loads and there are no shunt capacitors injecting reactive power. Most distribution circuits do not have large distributed generation, but shunt capacitors are turned on in peak load hours to regulate the voltage. Condition (2) removes the restriction on reactive power flow, but adds a new restriction on the ratio r/x. Condition (2) requires the ratio r/x to be non-decreasing along the line. Fortunately, this restriction is satisfied in distribution circuits. Interested readers can check that r/x is non-decreasing along the line in the numerical example in Figure 2 (data given in Table II).

Condition (2) can be used as rule of the thumb to decide the location of distributed generations, so that problems LMP and RLMP are equivalent. To demonstrate this, we give a numerical example. Figure 2 shows a 47-bus distribution circuit [25], which models an industrial feeder owned by the utility company Southern California Edison, with high penetration of photovoltaic (PV) generation. Bus 1 indicates the substation, and there are 5 photovoltaic (PV) generators located at bus 13, 17, 19, 23 and 24. The network data, including line impedances, peak MVA demand of loads, and the nameplate capacity of the shunt capacitors and the photovoltaic generations are listed in Table II.

To check Condition (2) in Theorem 1, we need to fix p. For simplicity, we assume that every load is absorbing its peak MVA demand at power factor 1, and every PV generator is generating power at its nameplate capacity. For instance, load at bus 22 is absorbing 2.23MW real power ($p_{22} = 2.23$MW), PV generator at bus 24 is generating 2MW real power ($p_{24} = -2$MW). Noting that line $(20, 21)$ has zero resistance and zero reactance, we can think of bus 21 and 24 as a single bus. Similarly, some other pair of buses can be thought of as a single bus (such as bus 2 and 13, 16 and 17, 18 and 19, etc.).

It is easy to compute that the nominal power flow P^{nom} is negative on some of the transmission lines. For example, the

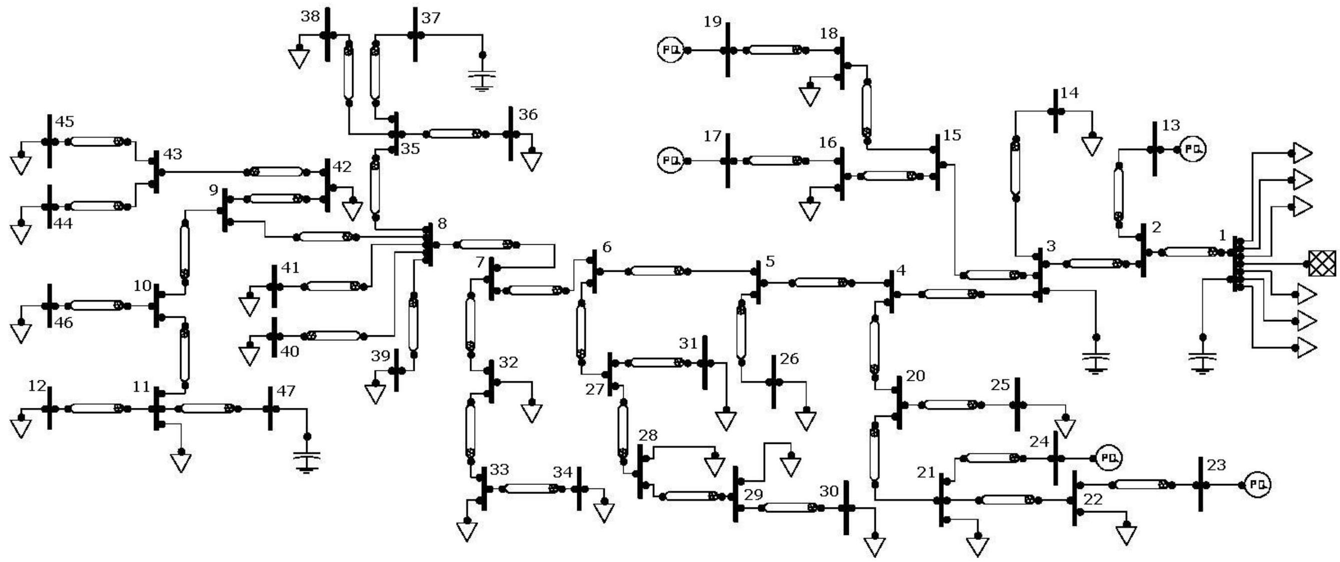

Fig. 2: Schematic diagram of a distribution feeder with high penetration of distributed generation (Photovoltaics). Bus No. 1 is the substation and the 6 loads attached to it model other feeders on this substation.

TABLE II: Network of Figure 2: Line impedances, peak spot load KVA, Capacitors and PV generation's nameplate ratings.

Network Data																	
Line Data				Line Data				Line Data				Load Data		Load Data		PV Generators	
From Bus.	To Bus.	R (Ω)	X (Ω)	From Bus.	To Bus.	R (Ω)	X (Ω)	From Bus.	To Bus.	R (Ω)	X (Ω)	Bus No.	Peak MVA	Bus No.	Peak MVA	Bus No.	Nameplate Capacity
1	2	0.259	0.808	8	41	0.107	0.031	21	22	0.198	0.046	1	30	34	0.2		
2	13	0	0	8	35	0.076	0.015	22	23	0	0	11	0.67	36	0.27	13	1.5MW
2	3	0.031	0.092	8	9	0.031	0.031	27	31	0.046	0.015	12	0.45	38	0.45	17	0.4MW
3	4	0.046	0.092	9	10	0.015	0.015	27	28	0.107	0.031	14	0.89	39	1.34	19	1.5 MW
3	14	0.092	0.031	9	42	0.153	0.046	28	29	0.107	0.031	16	0.07	40	0.13	23	1 MW
3	15	0.214	0.046	10	11	0.107	0.076	29	30	0.061	0.015	18	0.67	41	0.67	24	2 MW
4	20	0.336	0.061	10	46	0.229	0.122	32	33	0.046	0.015	21	0.45	42	0.13		
4	5	0.107	0.183	11	47	0.031	0.015	33	34	0.031	0	22	2.23	44	0.45	Shunt Capacitors	
5	26	0.061	0.015	11	12	0.076	0.046	35	36	0.076	0.015	25	0.45	45	0.2	Bus No.	Nameplate Capacity
5	6	0.015	0.031	15	18	0.046	0.015	35	37	0.076	0.046	26	0.2	46	0.45		
6	27	0.168	0.061	15	16	0.107	0.015	35	38	0.107	0.015	28	0.13				
6	7	0.031	0.046	16	17	0	0	42	43	0.061	0.015	29	0.13	Base Voltage (KV) = 12.35		1	6000 KVAR
7	32	0.076	0.015	18	19	0	0	43	44	0.061	0.015	30	0.2	Base KVA = 1000		3	1200 KVAR
7	8	0.015	0.015	20	21	0.122	0.092	43	45	0.061	0.015	31	0.07	Substation Voltage = 12.35		37	1800 KVAR
8	40	0.046	0.015	20	25	0.214	0.046					32	0.13			47	1800 KVAR
8	39	0.244	0.046	21	24	0	0					33	0.27				

nominal power flow $P_{20,21}^{nom}$ from bus 20 to bus 21 is given by

$$\begin{aligned} P_{20,21}^{nom} &= p_{21} + p_{22} + p_{23} + p_{24} \\ &= 0.45 + 2.23 - 1 - 2 = -0.32MW < 0, \end{aligned}$$

which violates condition (2). However, if we move the PV generation at bus $17, 19, 24$ to bus $15, 3, 20$ respectively, then condition (2) holds,[2] and problems LMP and RLMP become equivalent.

Hence, condition (2) gives guidelines on where to place the PV generations so that problems LMP and RLMP are equivalent. Basically, whenever condition (2) is violated due to some PV generation causing reverse power flow, the PV generation is larger than its downstream loads. For example, the PV at bus 24 causes reverse power flow on line $(20, 21)$ since it generates 2MW power, but its downstream load[3] is only

$$\begin{aligned} & p_{21} + p_{22} + p_{23} \\ = \;& 0.45 + 2.23 - 1 = 1.68MW < 2MW. \end{aligned}$$

[2] $P_{ij}^{nom} \geq 0$ except for the lines (i, j) whose r_{ij} and x_{ij} are zero.

[3] Counted from bus 21 downwards since bus 21 and bus 24 are connected by a line with zero r and x, and effectively a single bus.

To avoid reverse power flow, we can move the PV generation upwards in the circuit towards the feeder, so that its downstream loads gets larger, and eventually absorb all the power that the PV generates. For example, when we move the PV generation at bus 24 to bus 20, 0.45MW load at bus 25 is added to its downstream load, making it

$$\begin{aligned} & p_{21} + p_{22} + p_{23} + p_{25} \\ = \;& 0.45 + 2.23 - 1 + 0.45 = 2.13MW > 2MW. \end{aligned}$$

Consequently,

$$P_{20,21}^{nom} = 2.13 - 2 = 0.13MW > 0,$$

and condition (2) becomes satisfied on line $(20, 21)$. Using the same method, we can see that if we move PV generation from bus $17, 19, 24$ to bus $15, 3, 20$, then all reverse power flow will be eliminated, and condition (2) will hold.

In summary, condition (2) holds widely in distribution circuits without distributed generations. When there is distributed generation, condition (2) offers a simple guideline for placing them such that Problem LMP and RLMP remain equivalent. When distributed generation gets so large that reverse power flow cannot be avoided by placing the generations wisely, condition (4) suggests using uniform transmission lines to keep the equivalence between Problem LMP and RLMP.

V. Conclusion

We studied second-order cone relaxation of the loss minimization problem in radial networks using branch flow model. We proved that the relaxation is exact when there are no upper bounds on the voltage magnitudes and one of some parallel conditions is satisfied. The simplest condition we give is that there is no reverse real power flow on the lines. This condition holds if distributed generation is smaller than its downstream loads, which is true in most distribution circuits. This condition also gives guidelines to place the distributed generations—place them close to the substation such that reverse real power flow can be avoided.

Appendix

Lemma 2: *If RLMP-1 is feasible, then there exists an optimal solution (ℓ^*, P^*, Q^*, v^*) to RLMP-1.*

Proof: The set $\mathcal{F}$ of feasible ℓ for problem RLMP-1 is closed, and lies in the non-negative orthant ($\mathcal{F} \subseteq \mathbb{R}^n_+$). Let $\hat{\ell} \in \mathcal{F}$, and consider the set $\mathcal{O} := \{\ell \in \mathcal{F} \ : \ r^T\ell \leq r^T\hat{\ell}\}$. The set $\mathcal{O}$ is closed and bounded since $r > 0$, hence a compact set. Define $\ell^* := \underset{\ell\in\mathcal{O}}{\text{argmin}}\ c^T\ell$, and it gives an optimal solution to problem RLMP-1. □

Recall that the Lagrangian for problem RLMP-1 is

$$L(\ell,\lambda,u) = \sum_{i=0}^{N-1} r_i\ell_i + \sum_{i=0}^{N-1}\lambda_i\left(\frac{P_i^2+Q_i^2}{v_i} - \ell_i\right) + \sum_{i=1}^{N} u_i\left(\underline{v}_i - v_i\right)$$

Since the partial derivative of L with respect to ℓ_j evaluated at (ℓ^*, λ^*, u^*) is zero, it follows that

$$r_j = \lambda_j^* - \sum_{i=0}^{j}\frac{2r_jP_i^* + 2x_jQ_i^*}{v_i^*}\lambda_i^* + \sum_{i=0}^{N-1}\frac{{P_i^*}^2+{Q_i^*}^2}{{v_i^*}^2}\frac{\partial v_i}{\partial \ell_j}\lambda_i^* + \sum_{i=1}^{N}u_i^*\frac{\partial v_i}{\partial \ell_j} \tag{20}$$

for $j = 0, \ldots, n-1$. We are now going to derive (based on (20)) that if either condition from (1) to (4) in Lemma 1 holds, then $\lambda^* > 0$. Define

$$\bar{r}_k := \sum_{i=0}^{k} r_i, \ \bar{x}_k := \sum_{i=0}^{k} x_i$$

for $k = 0, \ldots, N-1$, and note that

$$\frac{\partial P_i}{\partial \ell_j} = \begin{cases} r_j & i \leq j \\ 0 & i \geq j+1 \end{cases}, \quad \frac{\partial Q_i}{\partial \ell_j} = \begin{cases} x_j & i \leq j \\ 0 & i \geq j+1 \end{cases},$$

$$\frac{\partial v_i}{\partial \ell_j} = \begin{cases} -2r_j\bar{r}_{j-1} - r_j^2 - 2x_j\bar{x}_{j-1} - x_j^2 & i \geq j+1 \\ -2r_j\bar{r}_{i-1} - 2x_j\bar{x}_{i-1} & i \leq j \end{cases}.$$

For brevity, we assume that problem RLMP-1 is feasible, and drop the superscript "*" (which denotes the optimal solution to RLMP-1) if there should be no confusion hereafter. Lemma 3–6 show that $\lambda > 0$ under one of conditions (1)-(4) in Lemma 1 respectively.

Lemma 3: *If $P_i^{nom} \geq 0$, $Q_i^{nom} \geq 0$ for each $i = 0, \ldots, n-1$, then $\lambda > 0$.*

Proof: The fact that (follows from (13)–(16))

- P, Q, v are all affine functions of l and $\frac{\partial P_i}{\partial \ell_j} \geq 0$, $\frac{\partial Q_i}{\partial \ell_j} \geq 0$, $\frac{\partial v_i}{\partial \ell_j} \leq 0$ for any $0 \leq i,j \leq N$;
- $(P^{nom}, Q^{nom}, v^{nom})$ is the (P,Q,v) corresponding to $\ell = 0$;
- $\ell \geq 0$;

implies that $P \geq P^{nom} \geq 0$, $Q \geq Q^{nom} \geq 0$, and $v \leq v^{nom}$. Since $\lambda_i \geq 0$ and $u_i \geq 0$, it follows from (20) that $r_j \leq \lambda_j$ for $j = 0, \ldots, n-1$, which implies $\lambda > 0$. □

Lemma 4: *If $P_i^{nom} \geq 0$ for each $i = 0, \ldots, n-1$, and $\frac{r_i}{x_i} \leq \frac{r_{i+1}}{x_{i+1}}$ for $i = 0, \ldots, n-2$, then $\lambda > 0$.*

Proof: If $\lambda > 0$ does not hold, then the set $\{i \geq 0 \ : \ \lambda_i = 0\}$ is non-empty. Define $k := \min\{i \geq 0 \ : \ \lambda_i = 0\}$. Then, $k \geq 1$ since by substituting $j = 0$ into (20),

$$r_0 \leq \lambda_0\left[1 - \frac{2(r_0P_0 + x_0Q_0)}{v_0}\right] \leq \lambda_0\frac{v_1}{v_0} \ \Rightarrow \ \lambda_0 > 0.$$

Define $\eta_i := r_i/x_i$, and note that η_i is non-decreasing in i. It follows from (20) that

$$\frac{r_k}{x_k} = -\sum_{i=0}^{k-1}\frac{2\eta_kP_i + Q_i}{v_i}\lambda_i + \sum_{i=1}^{N-1}\frac{P_i^2+Q_i^2}{v_i^2}\lambda_i\frac{\partial v_i}{x_k\partial \ell_k} + \sum_{i=1}^{N}u_i\frac{\partial v_i}{x_k\partial \ell_k}, \tag{21}$$

$$\frac{r_{k-1}}{x_{k-1}} = \frac{\lambda_{k-1}}{x_{k-1}} - \sum_{i=0}^{k-1}\frac{2\eta_{k-1}P_i + Q_i}{v_i}\lambda_i + \sum_{i=1}^{N-1}\frac{P_i^2+Q_i^2}{v_i^2}\lambda_i\frac{\partial v_i}{x_{k-1}\partial \ell_{k-1}} + \sum_{i=1}^{N}u_i\frac{\partial v_i}{x_{k-1}\partial \ell_{k-1}}. \tag{22}$$

Since $\eta_k \geq \eta_{k-1} > 0$ and $P_i \geq P_i^{nom} \geq 0$, we have

$$-\sum_{i=0}^{k-1}\frac{2\eta_kP_i + Q_i}{v_i}\lambda_i \leq -\sum_{i=0}^{k-1}\frac{2\eta_{k-1}P_i + Q_i}{v_i}\lambda_i. \tag{23}$$

We check that

$$\frac{\partial v_i}{x_k\partial \ell_k} \leq \frac{\partial v_i}{x_{k-1}\partial \ell_{k-1}}$$

for $i = 0, \ldots, N-1$. Hence,

$$\sum_{i=1}^{N-1}\frac{P_i^2+Q_i^2}{v_i^2}\lambda_i\frac{\partial v_i}{x_k\partial \ell_k} + \sum_{i=1}^{N}u_i\frac{\partial v_i}{x_k\partial \ell_k}, \leq \sum_{i=1}^{N-1}\frac{P_i^2+Q_i^2}{v_i^2}\lambda_i\frac{\partial v_i}{x_{k-1}\partial \ell_{k-1}} + \sum_{i=1}^{N}u_i\frac{\partial v_i}{x_{k-1}\partial \ell_{k-1}}.$$

Then, it follows from (21) and (22) that

$$\frac{r_k}{x_k} \leq \frac{r_{k-1}}{x_{k-1}} - \frac{\lambda_{k-1}}{x_{k-1}} < \frac{r_{k-1}}{x_{k-1}},$$

which contradicts with the condition $\frac{r_k}{x_k} \geq \frac{r_{k-1}}{x_{k-1}}$. Hence, we must have $\lambda > 0$. □

Lemma 5: *If $Q_i^{nom} \geq 0$ for each $i = 0, \ldots, n-1$, and $\frac{r_i}{x_i} \geq \frac{r_{i+1}}{x_{i+1}}$ for $i = 0, \ldots, n-2$, then $\lambda > 0$.*

Proof: Divide both sides of (20) by r_j instead of x_j for $j = k, k-1$ then the rest of the proof is follows that of Lemma 4. □

Lemma 6: *If* $\frac{x_i}{r_i} = \frac{x_j}{r_j}$ *for any* $0 \le i, j \le n-1$, *then* $\lambda > 0$.

Proof: If $\frac{x_i}{r_i} = \frac{x_j}{r_j}$ for any $0 \le i, j \le n-1$, then strictly equality holds in Equation (23). Therefore we can apply the proof of Lemma 4 to show that $\lambda > 0$. For interested readers, we present another proof here. Write (20) in a compact form (combining $j = 0, \ldots, n-1$) to obtain

$$r = \lambda - A\lambda - BCN\lambda - Bu, \tag{24}$$

where

$$A = \begin{pmatrix} \frac{2(r_0 P_0 + x_0 Q_0)}{v_0} & & \\ \vdots & \ddots & \\ \frac{2(r_{n-1} P_0 + x_{n-1} Q_0)}{v_0} & \cdots & \frac{2(r_{n-1} P_{n-1} + x_{n-1} Q_{n-1})}{v_{n-1}} \end{pmatrix},$$

$$B = -\begin{pmatrix} \frac{\partial v_1}{\partial \ell_0} & \cdots & \frac{\partial v_n}{\partial \ell_0} \\ \vdots & \ddots & \vdots \\ \frac{\partial v_1}{\partial \ell_{n-1}} & \cdots & \frac{\partial v_n}{\partial \ell_{n-1}} \end{pmatrix}, \quad N = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{P_1^2 + Q_1^2}{v_1^2} & & \\ & \ddots & \\ & & \frac{P_n^2 + Q_n^2}{v_n^2} \end{pmatrix}.$$

Define

$$w := CN\lambda + u \ge 0, \; \eta := \frac{x_0}{r_0},$$

and note that $\eta = \frac{x_i}{r_i}$ for all i. Left multiply both sides of (24) by the diagonal matrix R^{-1}, and note that $v_i - 2(r_i P_i + x_i Q_i) + (r_i^2 + x_i^2)\ell_i = v_{i+1}$, we have

$$1 = LR^{-1}\lambda - E\lambda - R^{-1}Bw,$$

where

$$L = \begin{pmatrix} \frac{v_1}{v_0} & & & \\ \frac{v_1 - v_0}{v_0} & \frac{v_2}{v_1} & & \\ \vdots & \vdots & \ddots & \\ \frac{v_1 - v_0}{v_0} & \frac{v_2 - v_1}{v_1} & \cdots & \frac{v_n}{v_{n-1}} \end{pmatrix},$$

$$E = (1+\eta^2) \begin{pmatrix} \frac{r_0 \ell_0}{v_0} & & & \\ \frac{r_0 \ell_0}{v_0} & \frac{r_1 \ell_1}{v_1} & & \\ \vdots & \vdots & \ddots & \\ \frac{r_0 \ell_0}{v_0} & \frac{r_1 \ell_1}{v_1} & \cdots & \frac{r_{n-1} \ell_{n-1}}{v_{n-1}} \end{pmatrix}.$$

Hence,

$$R^{-1}\lambda = L^{-1}1 + L^{-1}E\lambda + L^{-1}R^{-1}Bw. \tag{25}$$

By Claims 1-3 proven below, it follows from (25) that $R^{-1}\lambda > 0$. Consequently, $\lambda = R(R^{-1}\lambda) > 0$. □

We first give an important lemma. Define

$$J := \begin{pmatrix} 1 & & \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{pmatrix},$$

$$\hat{L} := \begin{pmatrix} a_1 & & & \\ a_1 - \Delta_1 & a_2 & & \\ \vdots & \vdots & \ddots & \\ a_1 - \Delta_1 & a_2 - \Delta_2 & \cdots & a_n \end{pmatrix},$$

where $a_i \neq 0$, $\Delta_i \in \mathbb{R}$ for $i = 1, \ldots, n$.

Lemma 7: *The matrix* $\hat{D} := \hat{L}^{-1}J$ *is given by*

$$\hat{D}_{ij} = \frac{1}{a_i} \prod_{k=j}^{i-1} \frac{\Delta_k}{a_k}.$$

Proof: Lemma 7 follows from Gaussian elimination. □

It follows from Lemma 7 that

$$D := L^{-1}J = \begin{pmatrix} \frac{v_0}{v_1} & & & \\ \frac{v_0}{v_2} & \frac{v_1}{v_2} & & \\ \vdots & \vdots & \ddots & \\ \frac{v_0}{v_n} & \frac{v_1}{v_n} & \cdots & \frac{v_{n-1}}{v_n} \end{pmatrix}.$$

Claim 1: $L^{-1}1 > 0$.

Proof: $L^{-1}1$ is the first column of the matrix D. Hence, it is point-wise positive. □

Claim 2: $L^{-1}E$ is point-wise nonnegative.

Proof: Since

$$L^{-1}E = (1+\eta^2)D \begin{pmatrix} \frac{r_0 \ell_0}{v_0} & & \\ & \ddots & \\ & & \frac{r_{n-1}\ell_{n-1}}{v_{n-1}} \end{pmatrix},$$

it is point-wise non-negative. □

Claim 3: $L^{-1}R^{-1}B$ is point-wise nonnegative.

Proof: Since $L^{-1}R^{-1}B = (1+\eta^2)L^{-1}J(I+N)RJ^T = (1+\eta^2)D(I+N)RJ^T$, it is point-wise non-negative. □

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