Quadratically constrained quadratic programs on acyclic graphs with application to power flow

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***Abstract*—We prove that non-convex quadratically constrained quadratic programs can be solved in polynomial time when their underlying graph is acyclic, provided the constraints satisfy a certain technical condition. When this condition is not satisfied, we propose a heuristic to obtain a feasible point. We demonstrate this approach on optimal power flow problems over radial networks.**

***Index Terms*—Quadratic programs, conic relaxations, optimal power flow, distribution networks.**

I. Introduction

A quadratically constrained quadratic program (QCQP) is an optimization problem in which the objective function and the constraints are quadratic. Many engineering problems can be represented as QCQPs, e.g., MIMO antenna beam-forming [1]–[4], sensor network localization [5], principal component analysis [6] and optimal power flow [7]–[9]. A wide-range of combinatorial problems can also be cast as QCQPs, e.g., the max-cut problem [10], [11] and the maximum stable set problem [12], [13]. In general, QCQPs are non-convex, and therefore lack computationally efficient solution methods. The contribution of this paper is to expand the class of non-convex QCQPs for which globally optimal solutions can be guaranteed.

The standard approach in the literature to solving a QCQP, optimally or approximately, is to relax this non-convex problem to a convex conic program [14], [15]. There are polynomial-time interior-point algorithms to solve these relaxed programs cast as second-order cone programs (SOCP) or semidefinite programs (SDP) [16]–[18]. For applications of this technique to engineering problems, we refer the reader to [14], [19]. Several authors have investigated the accuracy of these relaxations [10], [20]–[23]. Others have studied conditions under which a conic relaxation of the QCQP *is exact*, i.e., an optimal solution of the QCQP can be computed from an optimal solution of its relaxation [24], [25]. In Section II, we extend such results by proving a sufficient condition under which QCQPs with complex variables whose underlying graph structures are acyclic admit an efficient polynomial time solution through an SOCP or SDP relaxation. A special case of the main result was reported in [26] using the Lagrangian dual approach [14], [15]. Also, a similar result has been recently proved in [27] using an alternate method.

We apply the theory developed here to the optimal power flow (OPF) problem on radial networks in Section III. Originally formulated by Carpentier in 1962 [28], OPF seeks to minimize some cost function, such as power loss, generation cost and/or user utilities, subject to engineering constraints on a power network. As shown in [7]–[9], [29], [30], OPF can be formulated as a QCQP. We characterize a class of OPF problems over radial networks that have exact conic relaxations; the sufficient conditions in [31], [32], [33, Theorem 2], [34, Theorem 7] are special cases of this set.

When a QCQP does not satisfy the sufficient conditions for an exact relaxation, the convex relaxed problem may fail to deliver an optimal solution for the original non-convex problem. For such cases, we provide a heuristic approach in Section IV to obtain a feasible solution starting from the solution obtained from the relaxed problem. Through simulations, we demonstrate that this technique converges to a near-optimal feasible solution for OPF. We present a conclusion in Section V.

II. QCQP and relaxations

Consider the following QCQP with complex variable $x \in \mathbb{C}^n$, where $\mathbb{C}$ is the set of complex numbers.

Primal problem P:

$$\begin{aligned} \underset{x\in\mathbb{C}^n}{\text{minimize}} \quad & x^{\mathcal{H}} C_0 x \\ \text{subject to:} \quad & x^{\mathcal{H}} C_p x \le b_k, \quad p = 1, \dots, m, \end{aligned}$$

where $x^{\mathcal{H}}$ denotes the conjugate transpose of x, $C_0, \dots, C_m$ are $n \times n$ complex Hermitian matrices and $b_1, \dots b_m$ are scalars. If the matrices $C_0, \dots, C_m$ are positive semidefinite, then problem P is a convex program and can be solved in polynomial time [15], [35]. Otherwise, problem P is non-convex and NP-hard in general.

We next define

$$\mathcal{C} := \{C_0, C_1, \dots, C_m\}$$

and explore conditions on $\mathcal{C}$ that allow P to be solved in polynomial time. We start with some notation. Let $\mathbf{i} = \sqrt{-1}$. For any matrix H, let H_{jk} denote the entry in matrix H at the j-th row and k-th column. For any complex number z, let $Re\, z$ and $Im\, z$ denote the real and imaginary parts of z, respectively. Now consider the following problem with $n \times n$

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Hermitian matrix W.

Relaxed Problem RP**:**

$$\begin{aligned} \underset{W=W^{\mathcal{H}}}{\text{minimize}} \quad & \operatorname{tr}(C_0 W) \\ \text{subject to:} \quad & \operatorname{tr}(C_p W) \leq b_p, \quad p = 1, \dots, m, && (1) \\ & W \in \mathcal{W}, && (2) \end{aligned}$$

where the set of Hermitian matrices $\mathcal{W}$ satisfies the following property:

$$\text{For all } x \in \mathbb{C}^n, \quad xx^{\mathcal{H}} \in \mathcal{W}. \tag{3}$$

In other words, $\mathcal{W}$ contains all $n \times n$ positive semidefinite matrices of rank 1. RP is a relaxation of P because for any feasible solution x of problem P, $W = xx^{\mathcal{H}} \in \mathcal{W}$ is a feasible solution with the same objective function value of RP because $x^{\mathcal{H}} C_p x = \operatorname{tr}(C_p xx^{\mathcal{H}}) = \operatorname{tr}(C_p W)$ for all $p = 0, 1, \dots, m$. If there is an algorithm to solve RP in polynomial time and an optimal solution of P can be recovered from an optimal solution of RP in polynomial time, then P can be solved in polynomial time. RP is said to be an *exact relaxation* of P if an optimal solution for P can be computed from an optimal solution of RP. In what follows, we characterize the sets of matrices $\mathcal{C}$ and $\mathcal{W}$ such that the relaxation of P is exact and the optimal solution of P can be computed from the optimal solution of RP in polynomial time.

Suppose, there exists a function f defined on $n \times n$ Hermitian matrices that takes values in $\mathbb{C}^n$ and satisfies

$$\begin{gathered} \text{for all } W \in \mathcal{W},\ C \in \mathcal{C}, \\ \operatorname{tr}(CW) \geq x^{\mathcal{H}} C x, \text{ where } x = f(W). \end{gathered} \tag{4}$$

Then an optimal solution of P can be obtained by applying f to an optimal solution of RP. We derive conditions on $\mathcal{C}$ and $\mathcal{W}$ for such a function f to exist; we also derive a polynomial time function f in the process.

Define the $n \times 1$ vector of angles χ such that for $1 \leq k \leq n$,

$$x_k := |x_k| e^{\mathbf{i}\chi_k},$$

where $|z|$ denotes the absolute value of a complex number z. All angles lie in the interval $[0, 2\pi)$ and all angle operations are taken modulo 2π. To define the function f, we need to define $|x_k|$ and χ_k for $1 \leq k \leq n$. For $C \in \mathcal{C}$ and $W \in \mathcal{W}$, we have $\operatorname{tr}(CW) = \sum_{j,k} C_{jk} W_{jk}$. Then $\operatorname{tr}(CW) \geq x^{\mathcal{H}} C x$ if and only if:

$$\sum_{j \neq k} C_{jk}(x_j^{\mathcal{H}} x_k - W_{jk}) + \sum_k C_{kk}(|x_k|^2 - W_{kk}) \leq 0. \tag{5}$$

We ensure that the second sum in (5) is zero by requiring for all $1 \leq k \leq n$, $W_{kk} \geq 0$ and defining

$$|x_k| := \sqrt{W_{kk}}. \tag{6}$$

To define f, it remains to determine the angles χ such that $x = f(W)$ satisfies (4). Equivalently, we find χ such that the first sum in (5) is nonpositive. Define an $n \times n$ matrix of angles λ that satisfies $\lambda_{kj} = -\lambda_{jk}$ and rewrite the first sum in (5) as follows.

$$\begin{aligned} \sum_{j \neq k} C_{jk}(x_j^{\mathcal{H}} x_k - W_{jk}) &= \underbrace{\sum_{j \neq k} C_{jk}(x_j^{\mathcal{H}} x_k - |x_j||x_k| e^{\mathbf{i}\lambda_{jk}})}_{:=T_1} \\ &\quad + \underbrace{\sum_{j \neq k} C_{jk}(|x_j||x_k| e^{\mathbf{i}\lambda_{jk}} - W_{jk})}_{:=T_2}. \end{aligned} \tag{7}$$

Now, we derive conditions on $\mathcal{C}$ and $\mathcal{W}$ such that there exists a matrix λ for which the terms T_1 and T_2 in (7) are 0 and nonpositive, respectively. First we identify conditions on λ such that $T_1 = 0$.

Let $G_{\mathcal{C}}$ be the undirected graph on n vertices, such that there is an edge (j, k) between the nodes j and k, $j \neq k$ if and only if $C_{jk} \neq 0$ for some $C \in \mathcal{C}$. Note that $G_{\mathcal{C}}$ does not contain any self-loops. In graph $G_{\mathcal{C}}$, define the length of a path from j to k, denoted by $(j = k_0, k_1), (k_1, k_2), \dots (k_{r-1}, k_r = k)$ w.r.t. λ as:

$$\sum_{\ell=0}^{r-1} \lambda_{k_\ell k_{\ell+1}}.$$

Similarly, define the length of a cycle w.r.t. λ.

Lemma 1. *If all non-empty cycles in graph $G_{\mathcal{C}}$ have zero length w.r.t. λ, then there exists χ such that for all edges (j, k) in $G_{\mathcal{C}}$,*

$$\chi_k - \chi_j = \lambda_{jk}. \tag{8}$$

Proof: For each connected component of graph $G_{\mathcal{C}}$, pick any vertex j in the graph, and set χ_j to 0. For any other vertex k in the same connected component, consider *any* path from node j to node k and set χ_k to the length of that path. Since all cycles have zero length, all paths from j to k have the same length. The χ computed in this way satisfies (8), which completes the proof of Lemma 1. ■

Lemma 2. *If $G_{\mathcal{C}}$ is acyclic, then there exists χ such that for all edges (j, k) in $G_{\mathcal{C}}$,*

$$\chi_k - \chi_j = \lambda_{jk}.$$

Proof: The construction in the proof of Lemma 1 works for acyclic graphs $G_{\mathcal{C}}$. ■

Lemmas 1 and 2 provide sufficient conditions under which there exists χ that satisfies the relation in (8), given λ. If $G_{\mathcal{C}}$ contains non-empty cycles and there is a cycle that does not have zero length w.r.t. λ, then it may not be possible to construct χ that satisfies (8). Henceforth, restrict attention to the case where $G_{\mathcal{C}}$ is acyclic.

Given the matrix λ and the vector χ defined through Lemma 2, we have

$$\begin{aligned} T_1 &= \sum_{j \neq k} C_{jk} \left(x_j^{\mathcal{H}} x_k - |x_j||x_k| e^{\mathbf{i}\lambda_{jk}} \right) \\ &= \sum_{(j,k) \in G_{\mathcal{C}}} C_{jk} \left(|x_j||x_k| e^{\mathbf{i}(\chi_k - \chi_j)} - |x_j||x_k| e^{\mathbf{i}\lambda_{jk}} \right) \\ &= 0. \end{aligned}$$

The proof in Lemma 2 (and Lemma 1) is constructive. Thus it remains to further characterize the sets $\mathcal{C}$ and $\mathcal{W}$ so that a matrix of angles λ exists for which $T_2 \leq 0$. We achieve this in Lemma 3.

Define the $n \times n$ matrices of angles ω and $\gamma(C)$, where for $1 \leq j, k \leq n$,

$$W_{jk} = |W_{jk}|e^{\mathbf{i}\omega_{jk}}, \quad C_{jk} = |C_{jk}|e^{\mathbf{i}\gamma_{jk}(C)} \tag{9}$$

Using $|x_k| = \sqrt{W_{kk}}$ in the expression of T_2 in (7), we have

$$\begin{aligned} T_2 &= \sum_{j \neq k} C_{jk}\big(|x_j||x_k|e^{\mathbf{i}\lambda_{jk}} - W_{jk}\big) \\ &= \sum_{j \neq k} C_{jk}\Big(\sqrt{W_{jj}W_{kk}}e^{\mathbf{i}\lambda_{jk}} - |W_{jk}|e^{\mathbf{i}\omega_{jk}}\Big). \end{aligned}$$

Lemma 3. *For any $1 \leq j, k \leq n$, $j \neq k$. If $W \in \mathcal{W}$ satisfies*

$$\sqrt{W_{jj}W_{kk}} \geq |W_{jk}|, \tag{10}$$

and the set $\mathcal{C}$ satisfies

$$\max_{C \in \mathcal{C}} \gamma_{jk}(C) - \min_{C \in \mathcal{C}} \gamma_{jk}(C) \ \leq \ \pi, \tag{11}$$

then there exists an angle λ_{jk} such that

$$\text{Re}\left[C_{jk}\Big(\sqrt{W_{jj}W_{kk}}e^{\mathbf{i}\lambda_{jk}} - |W_{jk}|e^{\mathbf{i}\omega_{jk}}\Big)\right] \leq 0. \tag{12}$$

Proof: If $C_{jk} = 0$, the relation in (12) is trivially satisfied for arbitrary λ_{jk}. Assume $C_{jk} \neq 0$ henceforth. Define

$$a_{jk} := \Big(\sqrt{W_{jj}W_{kk}}e^{\mathbf{i}\lambda_{jk}} - |W_{jk}|e^{\mathbf{i}\omega_{jk}}\Big), \tag{13}$$

and let α_{jk} be the angle of a_{jk}, i.e., $a_{jk} = |a_{jk}|e^{i\alpha_{jk}}$. Using (13), the left-hand side of the relation in (12) becomes

$$\mathit{Re}[C_{jk} \ a_{jk}] = |C_{jk}||a_{jk}| \cos(\gamma_{jk}(C) + \alpha_{jk}).$$

The right-hand side of the above equation is nonpositive if for all $C \in \mathcal{C}$,

$$\cos(\gamma_{jk}(C) + \alpha_{jk}) \leq 0. \tag{14}$$

Since $\cos(\cdot)$ is nonpositive in $[\pi/2, 3\pi/2]$, the inequality in (14) holds if

$$\frac{\pi}{2} - \min_{C \in \mathcal{C}} \gamma_{jk}(C) \ \leq \ \alpha_{jk} \ \leq \ \frac{3\pi}{2} - \max_{C \in \mathcal{C}} \gamma_{jk}(C). \tag{15}$$

Such an α_{jk} exists provided $\frac{3\pi}{2} - \max_{C \in \mathcal{C}} \gamma_{jk}(C) \geq \frac{\pi}{2} - \min_{C \in \mathcal{C}} \gamma_{jk}(C)$, which is equivalent to the condition in (11). Now, pick *any* α_{jk} that satisfies (15). For this choice of α_{jk}, we are only left with computing the corresponding λ_{jk}.

From (13), it follows that $\mathit{Im}[a_{jk}e^{-\mathbf{i}\alpha_{jk}}] = 0$. Thus, we have

$$\sin(\lambda_{jk} - \alpha_{jk}) = \frac{|W_{jk}|}{\sqrt{W_{jj}W_{kk}}} \sin(\omega_{jk} - \alpha_{jk}). \tag{16}$$

The above equation always has a solution for λ_{jk} if the term on the right-hand side of (16) lies in the interval $[-1, +1]$, which is guaranteed if $W \in \mathcal{W}$ satisfies (10). This completes the proof of Lemma 3. ∎

Here, we interpret the conditions required for $\mathcal{W}$ and $\mathcal{C}$. The relation in (10) is equivalent to:

$$W_{kk} \geq 0 \qquad \text{and} \qquad W_{jj}W_{kk} \geq |W_{jk}|^2,$$

for each $1 \leq j, k \leq n$, $j \neq k$. This condition is satisfied if $\mathcal{W}$ is the set of all $n \times n$ Hermitian matrices with positive semidefinite 2×2 principal minors, which is also satisfied if the set $\mathcal{W}$ is the set of all $n \times n$ positive semidefinite matrices. These sets are convex and the relaxation RP can be solved as a conic program [14], [15], [24].

To interpret the relation in (11), consider the complex numbers C_{jk} for the matrices $C \in \mathcal{C}$. The corresponding angles $\gamma_{jk}(C), C \in \mathcal{C}$, satisfy (11), if there exists a line through the origin such that all the complex numbers C_{jk}, $C \in \mathcal{C}$, lie on one side of this line; see Figure 1 for examples.

Note that the proof is constructive and hence the angle λ_{jk} can be computed for all $1 \leq j, k \leq n$, $j \neq k$. Also, if λ_{jk} satisfies (12), then $\lambda_{kj} = -\lambda_{jk}$ is such that

$$\mathit{Re}\left[C_{kj}\Big(\sqrt{W_{kk}W_{jj}}e^{\mathbf{i}\lambda_{kj}} - |W_{kj}|e^{\mathbf{i}\omega_{kj}}\Big)\right] \leq 0,$$

and Lemma 3 provides the construction of a matrix of angles λ that satisfies the condition in Lemma 2. This completes the characterization of the sets $\mathcal{C}$ and $\mathcal{W}$ and the construction of f for which (4) holds.

To recapitulate, we delineate the construction of f. For each edge (j, k) in $G_{\mathcal{C}}$, pick α_{jk} that satisfies (15). For this α_{jk}, compute λ_{jk} from (16). Set $\lambda_{kj} = -\lambda_{jk}$. Note that we only construct λ_{jk} corresponding to the edges (j, k) in $G_{\mathcal{C}}$. This is enough to define the angles χ using Lemma 2. Using this construction, define $x = f(W)$ where $x_k = \sqrt{W_{kk}} \ e^{\mathbf{i}\chi_k}$. Thus we have proved the following result.

Theorem 4. *Suppose $\mathcal{C}$ and $\mathcal{W}$ satisfy the following:*

1) *$G_{\mathcal{C}}$ is acyclic,*
2) *For each edge (j, k) in $G_{\mathcal{C}}$,*

$$\begin{aligned} \max_{C \in \mathcal{C}} \gamma_{jk}(C) - \min_{C \in \mathcal{C}} \gamma_{jk}(C) &\leq \pi, \\ \sqrt{W_{jj}W_{kk}} &\geq |W_{jk}|. \end{aligned}$$

Then for all $W \in \mathcal{W}$ and $C \in \mathcal{C}$,

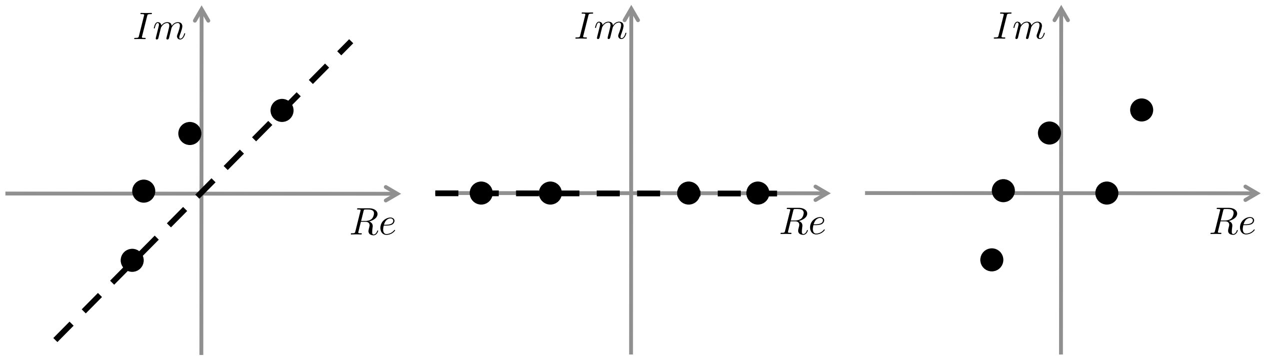
$$\mathrm{tr}(CW) \geq [f(W)]^{\mathcal{H}} \ C \ [f(W)].$$

RP can be solved over the set of $n \times n$ Hermitian matrices such that 2×2 principal minor corresponding to every edge (j, k) in the acyclic graph $G_{\mathcal{C}}$ is positive semidefinite. This constraint is a second-order cone constraint [15] and hence can be solved as an SOCP that is polynomial time computable [16]–[18]. RP can also be solved over $\mathcal{W}$ being the set of $n \times n$ positive semidefinite matrices and this is an SDP [14], [15]. From Theorem 4, it follows that for any optimal solution W_* of the SOCP or SDP relaxation of P, $x_* = f(W_*)$ is an optimal solution of P and hence the SOCP or SDP relaxation of P is exact. This result has been independently proved recently in [27].

A. QCQP in the real domain

Suppose in the QCQP P, the matrices in set $\mathcal{C}$ are real and symmetric, then (11) is always satisfied. If in addition, the graph $G_{\mathcal{C}}$ is acyclic, then Theorem 4 implies that the SDP or the SOCP relaxation RP is exact and thus we can obtain an optimal solution $x_* \in \mathbb{C}^n$ of P in polynomial time.

Let $\mathbb{R}$ denote the set of real numbers. Many authors [24], [25] have considered the case where P is solved over $x \in \mathbb{R}^n$



(a) (b) (c)

Fig. 1: (a) and (b) are examples of sets of complex numbers whose angles satisfy (11). (c) is an example of a set whose angles that do not.

and RP is solved over a real symmetric matrix $W \in \mathbb{R}^{n\times n}$. Here, we provide a sketch of how this result follows from the relation in (7).

For such a real QCQP, the angles $\gamma_{jk}(C), \omega_{jk}, \lambda_{jk}, \alpha_{jk}$ for edge (j,k) in the graph $G_{\mathcal{C}}$, $C \in \mathcal{C}$ and χ_k for $1 \leq k \leq n$ are restricted to be in the set $\{0, \pi\}$. Consider the additional constraint that for each (j,k) in $G_{\mathcal{C}}$, $\gamma_{jk}(C)$ for all $C \in \mathcal{C}$ is either 0 or π, i.e., the real numbers $[C_p]_{jk}$ for all $p = 0, 1, \ldots, m$ have the same sign. Choose λ_{jk} as follows:

$$\lambda_{jk} := \begin{cases} \pi, & \text{if } [C_p]_{jk} \geq 0 \quad \text{for } p = 0, 1, \ldots, m, \\ 0, & \text{if otherwise} \quad \text{for } p = 0, 1, \ldots, m. \end{cases}$$

From (12), these angles satisfy $T_2 \leq 0$. Thus, we are only left to prove that there exists an $n \times 1$ vector of angles $\chi \in \{0, \pi\}^n$ such that $\chi_k - \chi_j = \lambda_{jk}$ and hence $T_1 = 0$. It can be checked that this is equivalent to the *uniformly almost off-diagonal nonpositive* condition in [25], i.e., there exists $\sigma \in \{-1, 1\}^n$ such that for each edge $(j,k) \in G_{\mathcal{C}}$, we have $[C_p]_{jk}\sigma_j\sigma_k \leq 0$ for $p = 0, \ldots, m$. This essentially proves the result in [25, Theorem 3.4].

Note that in the sketch provided, $G_{\mathcal{C}}$ may contain cycles. Corresponding to each edge $(j,k) \in G_{\mathcal{C}}$, there is a sign (+ or $-$) of the entries $[C_p]_{jk}, p = 0, 1, \ldots, m$. The result requires the sign pattern of the graph to satisfy the uniformly almost off-diagonal nonpositive condition.

Now restrict attention to QCQPs where $G_{\mathcal{C}}$ is acyclic. Then [25, Theorem 3.4] and Theorem 4 both imply that the QCQP in the real domain can be solved in polynomial time using SDP or SOCP relaxations. Theorem 4, however, generalizes the result in [25] to complex QCQPs and cannot be obtained by transforming a QCQP in the complex domain to an equivalent QCQP in the real domain using the following transformation [14], [15] of the quadratic forms.

$$x^{\mathcal{H}}Cx = \begin{pmatrix} Re\, x \\ Im\, x \end{pmatrix}^T \begin{pmatrix} Re\, C & -Im\, C \\ Im\, C & Re\, C \end{pmatrix} \begin{pmatrix} Re\, x \\ Im\, x \end{pmatrix},$$

where for any vector or matrix y, y^T denotes its transpose. This discussion is summarized in the following.

Corollary 5. *Suppose $C \in \mathbb{R}^{n\times n}$ for all $C \in \mathcal{C}$ and $G_{\mathcal{C}}$ is acyclic.*

1) *Then an optimal solution $x_* \in \mathbb{C}^n$ of P can be obtained in polynomial time.*
2) *If for each edge (j,k) in $G_{\mathcal{C}}$, the real numbers $C_{jk}, C \in \mathcal{C}$ have the same sign, then an optimal solution $x_* \in \mathbb{R}^n$ of P can be obtained in polynomial time.*

Remark 1. *The authors in [24], [25] consider an additional convex constraint in P of the form*

$$x^2 \in \mathcal{F},$$

where x^2 is the $n \times 1$ vector with $(x_i)^2$ as its ith component. This adds the constraint

$$\mathrm{diag}(W) \in \mathcal{F},$$

in the relaxation RP. Our proofs remain unchanged with this additional constraint on the diagonal elements of W.

III. Optimal Power Flow: An application

In this section, we apply the results of Section II to the optimal power flow (OPF) problem. We start by summarizing some of the recent results on OPF relaxations in Section III-A. In Section III-B we formulate OPF as a QCQP. In Section III-C we restrict our attention to OPF over radial networks and use Theorem 4 to provide a sufficient condition under which OPF can be solved efficiently. Radial networks are important as most distribution systems are radial.

A. Prior work

As previously discussed, OPF can be cast as a QCQP. Various nonlinear programming techniques have been applied to the resulting non-convex problem, e.g., in [29], [30], [36]. An SDP based relaxation for OPF is proposed in [7], [8] and its use is illustrated on several IEEE test systems in [37] using an interior-point method. The authors in [9], [38] propose to solve the convex Lagrangian dual of the OPF problem and derive a sufficient condition that must be satisfied by the dual optimal solution for an optimal solution of OPF to be recovered from it.

Though an SDP relaxation recovers an OPF solution for most IEEE test systems, it does not work on all problem instances; such limitations have been most recently discussed in [39], though the non-convexity of power flow solutions has been studied earlier, e.g., in [40]–[43].

Recently a series of works have studied OPF over radial networks and proved a variety of sufficient conditions that guarantee exact convex relaxations. It has been independently reported in [31], [32], [43] that the semidefinite relaxation of OPF is exact for radial networks provided certain conditions on the power flow constraints are satisfied. A different approach to OPF has been explored using the branch flow model, first introduced in [44], [45]. While [46] studies a linear approximation of this model, various relaxations based on second-order cone programming (SOCP) have been studied in [47]–[50]. Authors in [48]–[50] prove that this relaxation is exact for radial networks when there are no upper bounds on loads, or when there are no upper bounds on voltage magnitudes.

Motivated by the results in [48], a more general branch flow model is introduced in [51] for the power flow analysis and optimization for both radial and meshed networks. The precise relationships between the SOCP based relaxations and the SDP based relaxations for OPF has been recently identified in [52].

B. Problem Formulation

Consider a power system network with n nodes (buses). The admittance-to-ground at bus i is y_{ii} and the admittance of the line between connected nodes i and j (denoted by $i \sim j$) is $y_{ij} = g_{ij} - \mathbf{i}b_{ij}$. Typically, $g_{ij} \geq 0$ and $b_{ij} \geq 0$, i.e., the lines are resistive and inductive. Define the corresponding $n \times n$ admittance matrix Y as

$$Y_{ij} = \begin{cases} y_{ii} + \sum_{j \sim i} y_{ij}, & \text{if } i = j, \\ -y_{ij}, & \text{if } i \neq j \text{ and } i \sim j, \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Remark 2. *Y is symmetric but not necessarily Hermitian.*

The remaining circuit parameters and their relations are defined as follows.

- V and I are n-dimensional complex voltage and current injection vectors, where V_k, I_k denote the nodal voltage and the injection current at bus $1 \leq k \leq n$ respectively. The voltage magnitude $|V_k|$ is bounded as
$$0 < \underline{W}_k \leq |V_k|^2 \leq \overline{W}_k.$$
- $S_k = P_k + \mathbf{i}Q_k$ is the complex power injection at node $1 \leq k \leq n$, where P_k and Q_k, respectively, denote the real and reactive power injections and
$$S_k = V_k I_k^{\mathcal{H}}. \tag{18}$$
- P_k^D and Q_k^D are the real and reactive power demands at bus $1 \leq k \leq n$, which are assumed to be fixed and given.
- P_k^G and Q_k^G are the real and reactive power generation at bus $1 \leq k \leq n$. They are decision variables that satisfy the constraints $\underline{P}_k^G \leq P_k^G \leq \overline{P}_k^G$ and $\underline{Q}_k^G \leq Q_k^G \leq \overline{Q}_k^G$.

Power balance at each bus $1 \leq k \leq n$ requires $P_k^G = P_k^D + P_k$ and $Q_k^G = Q_k^D + Q_k$, which leads us to define

$$\begin{aligned} \underline{P}_k &:= \underline{P}_k^G - P_k^D, & \overline{P}_k &:= \overline{P}_k^G - P_k^D \\ \underline{Q}_k &:= \underline{Q}_k^G - Q_k^D, & \overline{Q}_k &:= \overline{Q}_k^G - Q_k^D. \end{aligned}$$

The power injections at each bus $1 \leq k \leq n$ are then bounded as

$$\underline{P}_k \leq P_k \leq \overline{P}_k, \qquad \underline{Q}_k \leq Q_k \leq \overline{Q}_k.$$

The branch power flows and their limits are defined as follows.

- $S_{ij} = P_{ij} + \mathbf{i}Q_{ij}$ is the sending-end complex power flow from node i to node j, where P_{ij} and Q_{ij} are the real and reactive power flows respectively. The real power flows are constrained as $|P_{ij}| \leq \overline{F}_{ij}$ where $\overline{F}_{ij}$ is the line-flow limit between nodes i and j and $\overline{F}_{ij} = \overline{F}_{ji}$.
- $L_{ij} = P_{ij} + P_{ji}$ is the power loss over the line between nodes i and j, satisfying $L_{ij} \leq \overline{L}_{ij}$ where $\overline{L}_{ij}$ is the thermal line limit and $\overline{L}_{ij} = \overline{L}_{ji}$. Also, observe that since $L_{ij} \geq 0$, we have $|P_{ij}| \leq \overline{F}_{ij}, |P_{ji}| \leq \overline{F}_{ji}$ if and only if $P_{ij} \leq \overline{F}_{ij}, P_{ji} \leq \overline{F}_{ji}$.

For $1 \leq k \leq n$, let $J_k = e_k e_k^{\mathcal{H}}$ where e_k is the k-th canonical basis vector in $\mathbb{C}^n$. Define $Y_k := e_k e_k^{\mathcal{H}} Y$. Substituting these expressions into (18) yields

$$\begin{aligned} S_k &= e_k^{\mathcal{H}} V I^{\mathcal{H}} e_k = \operatorname{tr}\left(VV^{\mathcal{H}}(Y^{\mathcal{H}} e_k e_k^{\mathcal{H}})\right) = V^{\mathcal{H}} Y_k^{\mathcal{H}} V \\ &= \left(V^{\mathcal{H}} \underbrace{\left(\frac{Y_k^{\mathcal{H}} + Y_k}{2}\right)}_{=:\Phi_k} V \right) \\ &\qquad + \mathbf{i} \left(V^{\mathcal{H}} \underbrace{\left(\frac{Y_k^{\mathcal{H}} - Y_k}{2\mathbf{i}}\right)}_{=:\Psi_k} V \right), \end{aligned} \tag{19}$$

where Φ_k and Ψ_k are Hermitian matrices. Thus, the two quantities $V^{\mathcal{H}}\Phi_k V$ and $V^{\mathcal{H}}\Psi_k V$ are real numbers and

$$P_k = V^{\mathcal{H}}\Phi_k V, \quad Q_k = V^{\mathcal{H}}\Psi_k V.$$

The real power flow from i to j can be expressed as a quadratic form as follows.

$$P_{ij} = \mathit{Re}\{V_i(V_i - V_j)^{\mathcal{H}} y_{ij}^{\mathcal{H}}\} = V^{\mathcal{H}} M^{ij} V, \tag{20}$$

where M^{ij} is an $n \times n$ Hermitian matrix. Further details of these matrices are provided in the appendix.

The thermal loss of the line connecting buses i and j is

$$L_{ij} = L_{ji} = P_{ij} + P_{ji} = V^{\mathcal{H}} T^{ij} V \tag{21}$$

where $T^{ij} = T^{ji} := M^{ij} + M^{ji} \succeq 0$.

For a Hermitian $n \times n$ matrix C_0, we have

Optimal power flow problem OPF:

$$\underset{V \in \mathbb{C}^n}{\text{minimize}} \;\; V^{\mathcal{H}} C_0 V$$

subject to:

$$\underline{P}_k \leq V^{\mathcal{H}} \Phi_k V \leq \overline{P}_k, \quad 1 \leq k \leq n, \tag{22a}$$
$$\underline{Q}_k \leq V^{\mathcal{H}} \Psi_k V \leq \overline{Q}_k, \quad 1 \leq k \leq n, \tag{22b}$$
$$\underline{W}_k \leq V^{\mathcal{H}} J_k V \leq \overline{W}_k, \quad 1 \leq k \leq n, \tag{22c}$$
$$V^{\mathcal{H}} M^{ij} V \leq \overline{F}_{ij}, \quad i \sim j, \tag{22d}$$
$$V^{\mathcal{H}} T^{ij} V \leq \overline{L}_{ij}, \quad i \sim j, \tag{22e}$$

where (22a)–(22e) are, respectively, constraints on the real and reactive powers, the voltage magnitudes, the line flows and thermal losses.

We do not include line-flow constraints that impose an upper bound on the apparent power $\sqrt{P_{ij}^2 + Q_{ij}^2}$ on each branch $i \sim j$ because constraints are not quadratic in the voltages and hence beyond the scope of our model.

Remark 3 (Objective Functions)**.** *We consider different optimality criteria by changing C_0 as follows:*

- *Voltages: $C_0 = \mathcal{I}_{n\times n}$ (identity matrix) where we aim to minimize $\|V\|^2 = \sum_k |V_k|^2$.*
- *Power loss: $C_0 = (Y + Y^{\mathcal{H}})/2$ where we aim to minimize $\sum_i g_{ii}|V_i|^2 + \sum_{i<j} g_{ij}|V_i - V_j|^2$.*
- *Production costs: $C_0 = \sum_k c_k \Phi_k$ where we aim to minimize $\sum_k c_k P_k^G$, $c_k \geq 0$.*

C. Conic relaxation of OPF over radial networks

Assume hereafter that OPF is feasible. To conform to the notations of Section II, we replace the constraint in (22a) by the equivalent constraints

$$\begin{aligned} V^{\mathcal{H}}[\Phi_k]V &\leq \overline{P}_k, \quad 1 \leq k \leq n, \\ V^{\mathcal{H}}[-\Phi_k]V &\leq -\underline{P}_k, \quad 1 \leq k \leq n. \end{aligned}$$

Similarly we rewrite (22b) and (22c). Then the set of matrices $\{C_1, \ldots C_m\}$ and the set of scalars $\{b_1, \ldots, b_m\}$ in the OPF problem are defined as

$$\begin{aligned} &\{C_1, \ldots C_m\} \\ &\quad := \{\Phi_k, -\Phi_k, \Psi_k, -\Psi_k, J_k, -J_k, \ 1 \leq k \leq n\} \\ &\qquad \bigcup \ \left\{M^{ij}, T^{ij}, \ i \sim j\right\}, \end{aligned} \tag{23}$$

$$\begin{aligned} &\{b_1, \ldots, b_m\} \\ &\quad := \left\{\overline{P}_k, -\underline{P}_k, \overline{Q}_k, -\underline{Q}_k, \overline{W}_k, -\underline{W}_k, \ 1 \leq k \leq n\right\} \\ &\qquad \bigcup \ \left\{\overline{F}^{ij}, \overline{L}^{ij}, \ i \sim j\right\}. \end{aligned} \tag{24}$$

We limit the discussion to OPF instances where the graph of the power network is acyclic. Denote this graph on n nodes as $\mathcal{T}$. Then, it can be checked that for all objective functions considered, the set $\mathcal{C} = \{C_0, C_1, \ldots, C_m\}$ for OPF satisfies

$$G_{\mathcal{C}} = \mathcal{T}, \tag{25}$$

i.e., the sparsity pattern of the matrices in the set $\mathcal{C}$ follows the acyclic graph $\mathcal{T}$ of the power network. To explore the relation in (11) for OPF over $\mathcal{T}$, consider an edge (i,j) in $\mathcal{T}$. The admittance of the line joining buses i and j is $g_{ij} - \mathbf{i}b_{ij}$. Then the complex numbers $[C_p]_{ij}, p = 1, \ldots, m$ are given as (the computations are provided in the appendix):

(a) $[\Phi_i]_{ij} = -g_{ij}/2 + \mathbf{i}b_{ij}/2$,
(b) $[\Phi_j]_{ij} = -g_{ij}/2 - \mathbf{i}b_{ij}/2$,
(c) $[\Psi_i]_{ij} = -b_{ij}/2 - \mathbf{i}g_{ij}/2$,
(d) $[\Psi_j]_{ij} = -b_{ij}/2 + \mathbf{i}g_{ij}/2$,
(e) $[M^{ij}]_{ij} = -g_{ij}/2 + \mathbf{i}b_{ij}/2$,
(f) $[M^{ji}]_{ij} = -g_{ij}/2 - \mathbf{i}b_{ij}/2$,
(g) $[T^{ij}]_{ij} = [T^{ji}]_{ij} = -g_{ij}$.

For the objective functions considered, we have

(a) Voltages: $[C_0]_{ij} = 0$,
(b) Power loss: $[C_0]_{ij} = -g_{ij}$,
(c) Production costs: $[C_0]_{ij} = -g_{ij}(c_i + c_j)/2 + \mathbf{i}b_{ij}(c_i - c_j)/2$.

For the purpose of this discussion, consider power loss minimization as the objective, i.e., $[C_0]_{ij} = -g_{ij}$. Also, assume $g_{ij} > b_{ij} > 0$. We plot the non-zero (i,j)-th entries of the matrices in $\mathcal{C}$ on the complex plane in Figure 2 and label each point with its corresponding matrix. Clearly if we consider all the points in Figure 2, there does not exist a line through the origin such that all these points lie on one side of the line, i.e., the relation in (11) is not satisfied for the set $\mathcal{C}$.

To apply Theorem 4 to OPF, consider an index-set $\mathcal{M} \subseteq \{1, 2, \ldots, m\}$ such that the relation in (11) is satisfied for the set of matrices C_0 and $\{C_p, p \in \mathcal{M}\}$. This corresponds to removing certain inequalities in OPF, i.e., $b_p = +\infty$ for $p \in \{1, 2, \ldots, m\} \setminus \mathcal{M}$. For example, removing $-\Phi_j$ from the set $\{C_1, \ldots, C_m\}$ corresponds to setting $\underline{P}_j = -\infty$. Recall that for any matrix $C \in \mathcal{C}$, $\gamma_{ij}(C)$ is the angle of the complex number C_{ij}.

Theorem 6. *For OPF with $\tilde{\mathcal{C}} = \{C_0, C_p, p \in \mathcal{M}\}$ over an acyclic power network $\mathcal{T}$, suppose $\max_{C\in\tilde{\mathcal{C}}} \gamma_{ij}(C) - \min_{C\in\tilde{\mathcal{C}}} \gamma_{ij}(C) \leq \pi$ for $i \sim j$ in $\mathcal{T}$. Then the SOCP or SDP relaxation of OPF is exact.*

We explore, through examples, some constraint patterns for OPF over $\mathcal{T}$ where the SOCP or SDP relaxation of OPF is exact.

Example 1: In Figure 2, consider the (i,j)-th elements of the following set of matrices:

$$\left\{\Phi_i, \Phi_j, \Psi_i, \Psi_j, -\Psi_i, M^{ij}, M^{ji}, T^{ij} = T^{ji}, C_0\right\}.$$

This set of points lie on one side of the line passing through the origin and $[\Psi_i]_{ij}$ on the complex plane. With this set of points, associate a constraint pattern defined as follows. For any point in the diagram that is not a part of this set, the inequality associated with that matrix is removed from OPF. For example, the matrices $-\Phi_j$, $-\Phi_i$ and $-\Psi_j$ are removed, which leads to

$$\underline{P}_j = \underline{P}_i = \underline{Q}_j = -\infty. \tag{26}$$

This can be generalized to a constraint pattern over $\mathcal{T}$ by removing the lower bounds on the real powers at all nodes and the lower bounds on reactive powers at alternate nodes.

Example 2: Suppose $\underline{P}_k = \underline{Q}_k = -\infty$ for all nodes k in $\mathcal{T}$. This corresponds to considering points only on the left-half plane in Figure 2 for all edges (i,j) in $\mathcal{T}$. Clearly, (11) is satisfied in this case. In Figure 2, we assume $g_{ij} > b_{ij} > 0$. Regardless of the ordering between g_{ij} and b_{ij} for edges (i,j)

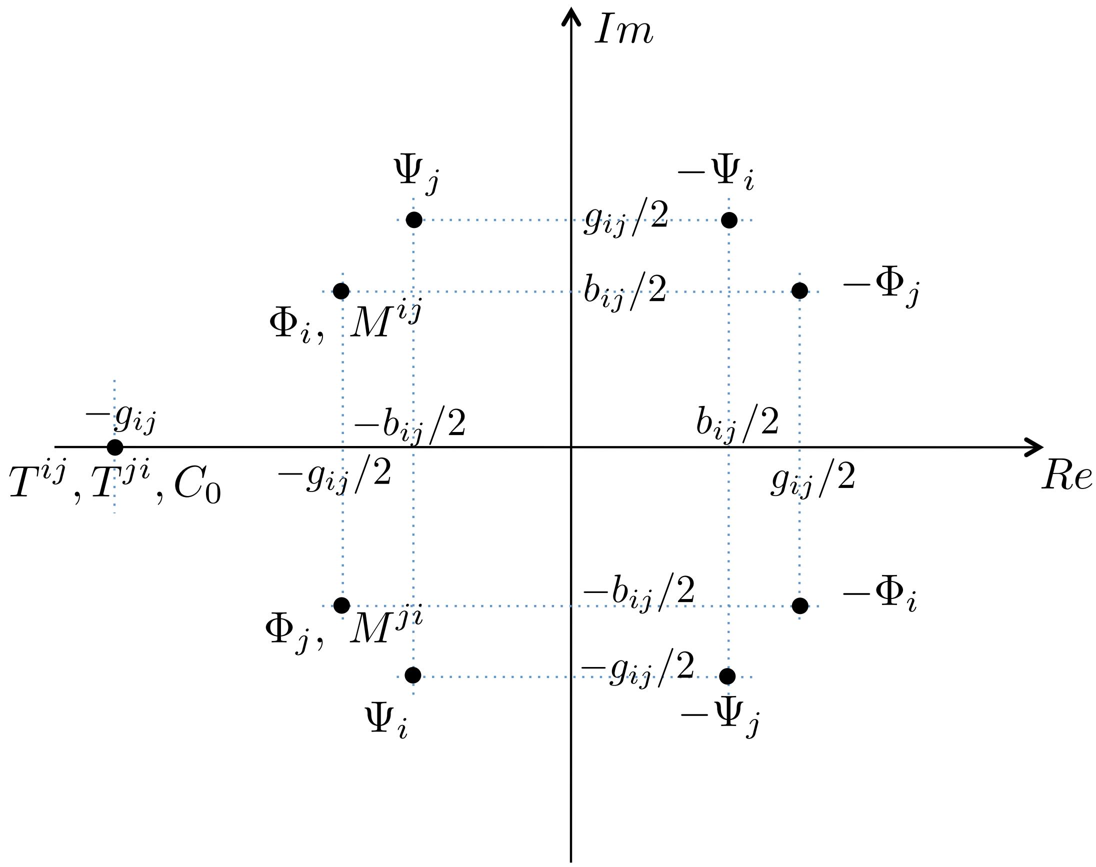


Fig. 2: C_{ij} and non-zero $[C_p]_{ij}, p = 1, \ldots m$ on the complex plane for OPF for a fixed line (i, j) in tree $\mathcal{T}$.

in $\mathcal{T}$, the set of points considered in this constraint pattern always lies in the left half of the complex plane.

Removing the lower bounds on the real and reactive power can be interpreted as load over-satisfaction, i.e., the real and reactive powers supplied to a node $1 \leq k \leq n$ can be greater than their respective real and reactive power demands P_k^D and Q_k^D. Results showing that OPF on a radial network with load over-satisfaction can be efficiently solved were previously reported in [31], [32], [43].

Example 3: Consider voltage minimization, i.e., $C_0 = \mathcal{I}_{n\times n}$. In Figure 2, consider the (i, j)-th entries of the following set of matrices:

$$\{-\Phi_i, \Phi_j, -\Phi_j, \Psi_i, -\Psi_j, C_0\}.$$

The constraint pattern associated with this set of points is

$$\overline{P}_i = \overline{Q}_j = \overline{L}_{ij} = \overline{L}_{ji} = \overline{F}_{ij} = \overline{F}_{ji} = +\infty, \text{ and } \underline{Q}_i = -\infty.$$

This set of constraints is consistent with (11) over the edge (i, j) and we can construct a constraint pattern for the OPF problem.

IV. Numerical examples

A. Numerical techniques

In Section II, we have identified conditions under which a QCQP problem P over an acyclic graph has an exact SOCP or SDP relaxation. When these sufficient conditions are not satisfied, P may not be solvable in polynomial time. For such a problem P, we now provide a heuristic approach to reach a feasible point of P, starting from the solution of its SOCP or SDP relaxation RP. For ease of presentation, we only consider the SDP relaxation in this section, but the method can be generally applied to other convex relaxations such as the SOCP based relaxation. Let W_* be an optimal solution of the SDP relaxation RP. If rank $W_* = 1$, then the spectral decomposition of $W_* = (x_*)(x_*)^{\mathcal{H}}$ yields an optimal solution x_* of P. If, however, rank $W_* > 1$, then $f(W_*)$ may not be a feasible point of P, because the set $\mathcal{C}$ for problem P may not satisfy the sufficient conditions in Theorem 4. In what follows, we propose a method to construct a feasible solution $\tilde{x}$ for P using such an optimal W_* of RP. The following relation characterizes the relationship between the optimal solution of P and its value at $\tilde{x}$.

$$\begin{aligned} &\text{objective value of } RP \text{ at } W_* \\ &\quad \leq \text{optimum objective value of } P \\ &\quad \leq \text{objective value of } P \text{ at } \tilde{x}. \end{aligned} \tag{27}$$

In many practical problems where rank $W_* > 1$, the principal eigenvalue of W_* is orders of magnitude greater than the other eigenvalues. We use the principal eigenvector to search for a "nearby" feasible point of P as follows. Let $w_* \in \mathbb{C}^n$ be the principal eigenvector of W_* and define the starting point $x[0]$

of the algorithm as

$$x[0] := w_* \sqrt{\mathrm{tr}(C_0 W_*)}.$$

This scaling ensures that the objective value at $x[0]$ is the optimum objective value of RP at W_*. An alternate starting point of the algorithm can be $x[0] := f(W_*)$. If $x[0]$ satisfies all constraints in P, then the algorithm ends with $\tilde{x} = x[0]$. Otherwise, we construct a sequence of points $(x[1], x[2], \ldots)$ where $x[r+1]$ is constructed from $x[r]$ as follows.

1) For $p = 1, 2, \ldots, m$, linearize the quadratic function $g_p(x) = x^{\mathcal{H}} C_p x$ around the point $x[r]$ and call this function $g_p^{(r)}(x)$, i.e.,

$$g_p^{(r)}(x) = x[r]^{\mathcal{H}} C_p x[r] + 2\,Re\left\{x[r]^{\mathcal{H}} C_p (x - x[r])\right\}.$$

2) For $p = 1, 2, \ldots, m$, define

$$s_p^{(r)}(x) := \begin{cases} g_p^{(r)}(x) - b_p, & \text{if } g_p^{(r)}(x) > b_p, \\ 0 & \text{otherwise,} \end{cases}$$

as the amount by which the linearized function $g_p^{(r)}$ violates the inequality constraint $g_p^{(r)}(x) \le b_p$.

3) Compute $x[r+1]$ using

$$\begin{aligned} x[r+1] = \arg\min_{x \in \mathbb{C}^n} \quad & \sum_{p=1}^{r} [s_p^{(r)}(x)]^2 \\ \text{subject to:} \quad & \|x - x[r]\|_1 \le \gamma, \end{aligned}$$

where $\|.\|_1$ denotes the ℓ^1 norm and γ is the maximum allowable step-size. This is a parameter for the algorithm and should be chosen such that the linearization $g_p^{(r)}$ is a reasonably good approximation of the quadratic form $g_p(x)$ for all $p = 1, 2, \ldots, m$ in the ℓ^1 ball centered around $x[r]$ with radius γ.

4) If $x[r+1]$ satisfies all of the constraints in P, then the algorithm ends with $\tilde{x} = x[r+1]$.

This heuristic approach either ends at a feasible point $\tilde{x}$ of P or it fails to produce one within a fixed number of iterations. In the next section, we show that this technique performs quite well for numerical OPF examples where the SDP relaxation yields an optimal solution W_* with rank more than 1.

B. OPF test examples

The SDP relaxation of OPF and the techniques described in Section III are illustrated on a sample distribution circuit from Southern California and randomly generated radial circuits. The SDP is solved in MATLAB using YALMIP [53]. For cases when the solution of the optimization RP yields a W_* such that rank $W_* = 1$ then the optimal voltage profile (V_*) of OPF is calculated using $W_* = (V_*)(V_*)^{\mathcal{H}}$. If the optimal W_* does not satisfy the rank condition, the heuristic approach described above is used to find a feasible point of the OPF. The feasible point obtained may not be optimal for the original problem, so we characterized its sub-optimality by defining the following quantity.

$$\eta := \frac{\text{Obj. value at heuristically reached feasible point}}{\text{Obj. value at optimal point of relaxed problem}} - 1.$$

Test system	SoCal distribution circuit		Random radial networks	
Minimize	Power-loss	Voltage	Power-loss	Voltage
rank W_*	1	≥ 1	1	≥ 1
Mean η	N/A	1.8%	N/A	0.5%
Maximum η	N/A	4.1%	N/A	1.5%

TABLE II: Summary of simulation results

Smaller values of η indicate that the feasible point obtained using the algorithm is close to the theoretical optimum of the OPF problem.

Throughout this section, let $y = (a, b)$ denote a random variable y drawn from a uniform distribution over the interval $[a, b]$. Using this notation, we describe the test systems used for the simulations.

1) *SoCal Distribution Circuit:* The sample industrial distribution system in Southern California has been previously described in [48]. It has a peak load of approximately 11.3 MW and installed PV generation capacity of 6.4MW. We modified this circuit by removing the 30MW load at the substation bus (this load represents other distribution circuits fed by the same substation) and simulated it with the parameters provided in Table I. To scale the problem correctly, all quantities were normalized to per unit (p.u.) quantities using the base values given in Table I.
2) *Random Test Circuits:* These circuits are generated using parameters typical of sparsely loaded rural circuits, as detailed in [54] and employed (with suitable modifications) in [55], [56]. Around 15-60% of the nodes are assumed to have 2 kW of PV capacity. The remaining parameters of these systems are described in Table I.

The tests are run with both voltage and power loss minimization as objective functions. The optimization results are summarized in Table II. For power loss minimization, we always obtain a rank 1 optimal W_*.

For voltage minimization, however, we obtain optimal solutions of RP that violate the rank condition. In these cases, the heuristic approach is used to find a feasible point of the OPF. We construct the solution based on the complex voltage $V_k = |V_k| e^{\mathbf{i}\theta_k}$ at bus $1 \le k \le n$. For the heuristic algorithm, define

$$x := (|V_2|, |V_3|, \ldots, |V_n|, \theta_2, \theta_3, \ldots, \theta_n),$$

and set the parameter $\gamma = +\infty$. In the examples studied, this approach always yields a feasible point within 5 iterations. Table II shows the mean and maximum values for η over the set of test cases performed. These results indicate that our algorithm generally finds a feasible point of the OPF with an objective value close to the theoretical optimum and hence performs well. A general bound on the performance of this heuristic technique, however, remains an open question.

V. Conclusion

QCQP problems are generally non-convex and NP-hard. This paper proves that a certain class of QCQP problems are solvable in polynomial-time. We have applied this result to the optimal power flow problem and derived a set of conditions

Test system	SoCal distribution circuit	Random radial networks
Number of nodes (n)	47	50-150
Line impedances $(y_{ij})^{-1}$	[48, Table 1]	$(0.33+\mathbf{i}0.38)\Omega/km$, length = $(0.2km, 0.3km)$
Voltage limits $\sqrt{\overline{W}_k}, \sqrt{\underline{W}_k}$	1 ± 0.05 p.u. at all nodes.	1 ± 0.05 p.u. at all nodes.
Real power demand P_k^D	[48, Table 1]	$(0, 4.5kW)$
Reactive power demand Q_k^D	Computed with $p.f. = (0.80, 0.98)$ lagging	$(0.2P_k^D, 0.3P_k^D)$
Real power gen. limits $\overline{P}_k^G, \underline{P}_k^G$	PV nodes: $\overline{P}_k^G = (0.2, 1.0)$ times capacity, Substation node: $\overline{P}_1^G = 10MW$. At all nodes, $\underline{P}_k^G = 0$.	PV nodes: $\overline{P}_k^G = (0, 2kW)$, Substation node: $\overline{P}_1^G$ scaled with n. At all nodes, $\underline{P}_k^G = 0$.
Reactive power gen. limits	$\overline{Q}_k^G = 0.3\overline{P}_k^G$, $\underline{Q}_k^G = -0.3\overline{P}_k^G$ at all nodes.	$\overline{Q}_k^G = 0.3\overline{P}_k^G$, $\underline{Q}_k^G = -0.3\overline{P}_k^G$ at all nodes.
Base quantities	$P_{base} = 1MW$, $V_{base} = 12.35kV(L-L)$.	$P_{base} = 1MW$, $V_{base} = 12.47kV(L-L)$.

TABLE I: Circuit parameters for semidefinite relaxation of OPF

under which this non-convex problem admits an efficient solution. For problems that do not satisfy our sufficient conditions, we provide a heuristic technique to find a feasible solution. Simulations suggest that this method often finds a near-optimal solution for the OPF problem.

VI. Appendix

A. Matrices involved in OPF:

Here we compute the (i,j)-th entries of the set of matrices $\{C_1, \ldots, C_m\}$ for OPF. Recall that the power network considered is $G_{\mathcal{C}} = \mathcal{T}$, which is acyclic by hypothesis. From (19), (20), (21), we have the following relations for $1 \leq k \leq n$, $p \sim q$ and $i \sim j$ in graph $\mathcal{T}$:

$$[\Phi_k]_{ij} = \begin{cases} \frac{1}{2}Y_{ij} = \frac{1}{2}(-g_{ij} + \mathbf{i}b_{ij}) & \text{if } k = i, \\ \frac{1}{2}Y_{ij}^{\mathcal{H}} = \frac{1}{2}(-g_{ij} - \mathbf{i}b_{ij}) & \text{if } k = j, \\ 0 & \text{otherwise,} \end{cases} \tag{28}$$

$$[\Psi_k]_{ij} = \begin{cases} \frac{-1}{2\mathbf{i}}Y_{ij} = \frac{1}{2}(-b_{ij} - \mathbf{i}g_{ij}) & \text{if } k = i, \\ \frac{1}{2\mathbf{i}}Y_{ij}^{\mathcal{H}} = \frac{1}{2}(-b_{ij} + \mathbf{i}g_{ij}) & \text{if } k = j, \\ 0 & \text{otherwise,} \end{cases} \tag{29}$$

$$[M^{pq}]_{ij} = \begin{cases} g_{pq} & \text{if } i = j = p \\ \frac{1}{2}(-g_{pq} + \mathbf{i}b_{pq}) & \text{if } (i,j) = (p,q), \\ \frac{1}{2}(-g_{pq} - \mathbf{i}b_{pq}) & \text{if } (i,j) = (q,p), \\ 0 & \text{otherwise,} \end{cases} \tag{30}$$

$$[T^{pq}]_{ij} = \begin{cases} g_{pq} & \text{if } i = j = p \text{ or } i = j = q, \\ -g_{pq} & \text{if } (i,j) = (p,q) \text{ or } (i,j) = (q,p), \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

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