Optimal Demand Response Based on Utility Maximization in Power Networks

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***Abstract*— Demand side management will be a key component of future smart grid that can help reduce peak load and adapt elastic demand to fluctuating generations. In this paper, we consider households that operate different appliances including PHEVs and batteries and propose a demand response approach based on utility maximization. Each appliance provides a certain benefit depending on the pattern or volume of power it consumes. Each household wishes to optimally schedule its power consumption so as to maximize its individual net benefit subject to various consumption and power flow constraints. We show that there exist time-varying prices that can align individual optimality with social optimality, i.e., under such prices, when the households selfishly optimize their own benefits, they automatically also maximize the social welfare. The utility company can thus use dynamic pricing to coordinate demand responses to the benefit of the overall system. We propose a distributed algorithm for the utility company and the customers to jointly compute this optimal prices and demand schedules. Finally, we present simulation results that illustrate several interesting properties of the proposed scheme.**

I. Introduction

Demand side management will be a key component of future smart grid that can help reduce peak load and adapt elastic demand to fluctuating generations. In this paper, we consider households that operate different appliances including PHEVs and batteries and propose a demand response approach based on utility maximization. Each appliance provides a certain benefit depending on the pattern or volume of power it consumes. Each household wishes to optimally schedule its power consumption so as to maximize its individual net benefit subject to various consumption and power flow constraints. We show that there exist time-varying prices that can align individual optimality with social optimality, i.e., under such prices, when the households selfishly optimize their own benefits, they automatically also maximize the social welfare. The utility company can thus use dynamic pricing to coordinate demand responses to the benefit of the overall system. We propose a distributed algorithm for the utility company and the customers to jointly compute this optimal prices and demand schedules. Finally, we present simulation results that illustrate several interesting properties of the proposed scheme, as follows.

First, different appliances are coordinated indirectly by real-time pricing, so as to flatten the total demand at different times as much as possible. Second, compared with no demand response or flat-price schemes, real-time pricing is very effective in shaping the demand: it not only greatly reduces the peak load, but also the variation in demand. Third, the integration of the battery helps reap more benefit from demand response: it does not only reduce the peak load but further flattens the entire profile and reduce the demand variation. Forth, the real-time pricing scheme can increase the load factor greatly and save a large amount of generation cost without hurting customers' utility; here again, battery amplifies this benefit. Fifth, the cost of battery (such as lifetime in terms of charging/discharging cycles) is important: the benefit of demand response increases with lower battery cost. Finally, as the number of the households increases, the benefit of our demand response increases but will eventually saturate.

There exists a large literature on demand response, see, e.g., [1], [2], [3], [4], [5], [6], [7], [8], [9], [10]. We briefly discuss some that are directly relevant to our paper. First there are papers on modeling specific appliances. For instance, [1] and [2] consider the electricity load control with thermal mass in buildings; [3] considers the coordination of charging PHEV with other electric appliances. Then, there are papers on the coordination among different appliances. [4] studies electricity usage for a typical household and proposes a method for customers to schedule their available distributed energy resources to maximize net benefits in a day-ahead market. [5] proposes a residential energy consumption scheduling framework which attempts to achieve a desired trade-off between minimizing the electricity payment and minimizing the waiting time for the operation of each appliance in household in presence of a real-time pricing tariff by doing price prediction based on prior knowledge. While in practice, for different appliances, the household may have a different objective than waiting time for the operation of the appliance.

Besides the work such as [4], [5] which considers a single household demand response given a pricing scheme, [6] considers a power network where end customers choose their daily schedules of their household appliances/loads by playing games among themselves and the utility company tries to adopt adequate pricing tariffs that differentiate the energy usage in time and level to make the Nash equilibrium minimize the energy costs. However, they assume that customers have full knowledge of generation cost function and in their proposed algorithm they require customers to update their energy consumption scheduling asynchronously, both of which are hard to implement in practice. [7] considers a centralized complex-bid market-clearing mechanism where customers submit price-sensitive bids in the day-ahead market, they did not study the specific electricity consumptions model for the household.

Notations. We use $q_{i,a}(t)$ to denote the power demanded by customer i for appliance a at time t. Then, $q_{i,a} := (q_{i,a}(t), \forall t)$ denotes the vector of power demands over $t = 1, \ldots, T$; $q_i := (q_{i,a}, \forall a \in \mathcal{A}_i)$ denotes the vector of power demands for all appliances in the collection $\mathcal{A}_i$ of customer i; and $q := (q_i, \forall i)$ denotes the vector of power demands from all customers. Similar convention is used for other quantities such as battery charging schedules $r_i(t), r_i, r$.

II. SYSTEM MODEL

Consider a set N of households/customers that are served by a single utility company. The utility company participates in wholesale markets (day-ahead, real-time balancing, ancillary services) to purchase electricity from generators and then sell it to the N customers in the retail market. Even though wholesale prices can fluctuate rapidly by large amounts, currently most utility companies hide this complexity and volatility from their customers and offer electricity at a flat rate (fixed unit price), perhaps in multiple tiers based on a customer's consumption. Even though the wholesale prices are determined by (scheduled or real-time) demand and supply and by congestion in the transmission network (except for electricity provisioned through long-term bilateral contracts), the retail prices are set statically independent of the real-time load and congestion. Flat-rate pricing has the important advantage of being simple and predictable, but it does not encourage efficient use of electricity. In this paper, we propose a way to use dynamic pricing in the retail market to coordinate the customers' demand responses to the benefit of individual customers and the overall system. We now present our model, describe how the utility should set their prices dynamically, how a customer should respond, and the properties of the resulting operating point.

We consider a discrete-time model with a finite horizon that models a day. Each day is divided into T timeslots of equal duration, indexed by $t \in \mathcal{T} := \{1, 2, \cdots, T\}$.

A. Utility company

The utility company serves as an intermediary that participates in multiple wholesale markets, including day-ahead, real-time balancing and ancillary services, to provision enough electricity to meet the demands of the N customers. The design of the retail prices needs to at least recover the running costs of the utility company, including the payments it incurs in the various wholesale markets. It is an interesting subject that is beyond the scope of this paper. For simplicity, we make the important assumption that this design can be summarized by a cost function $C(Q, t)$ that specifies the cost for the utility company to provide Q amount of power to the N customers at time t. The modeling of cost function is an active research issue [11], [10], [7]. Here we assume that the cost function $C(Q, t)$ is convex increasing in Q for each t. The utility company sets the prices $(p(t), t \in \mathcal{T})$ according to an algorithm described below.

B. Customers

Each customer $i \in N$ operates a set $\mathcal{A}_i$ of appliances such as air conditioner, refrigerator, plug-in hybrid electric vehicle (PHEV), etc. For each appliance $a \in \mathcal{A}_i$ of customer i, we denote by $q_{i,a}(t)$ its power draw at time $t \in \mathcal{T}$, and by $q_{i,a}$ the vector $(q_{i,a}(t), t \in \mathcal{T})$ of power draws over the whole day. An appliance a is characterized by two parameters:

- a utility function $U_{i,a}(q_{i,a})$ that quantifies the utility user i obtains when it consumes $q_{i,a}(t)$ power at each time $t \in \mathcal{T}$; and
- a set of linear inequalities $A^{i,a} q_{i,a} \le \eta_{i,a}$ on the vector power $q_{i,a}$.

In Section IV, we will describe in detail how we model various appliances through appropriate matrices $A^{i,a}$ and vector $\eta_{i,a}$. Note that inelastic load, e.g., minimum refrigerator power, can be modeled by $q_{i,a}(t) \ge \underline{q}_{i,a}$ that says the appliance a of customer i requires a minimum power $\underline{q}_{i,a}$ at all times t. This is a linear inequality constraint and part of $A^{i,a} q_{i,a} \le \eta_{i,a}$.

C. Energy storage

In addition to appliances, a customer i may also possess a battery which provides further flexibility for optimization of its consumption across time. We denote by B_i the battery capacity, by $b_i(t)$ the energy level of the battery at time t, and by $r_i(t)$ the power (energy per period) charged to (when $r_i(t) \ge 0$) or discharged from (when $r_i(t) < 0$) the battery at time t. Assume that battery power leakage is negligible. Then we model the dynamics of the battery energy level by

$$b_i(t) \quad = \quad \sum_{\tau=1}^{t} r_i(\tau) + b_i(0) \tag{1}$$

Battery usually has an upper bound on charge rate, denoted by r_i^{max} for customer i, and an upper bound on discharge rate, denoted by $-r_i^{min}$ for customer i. We thus have the following constraints on $b_i(t)$ and $r_i(t)$:

$$0 \le b_i(t) \le B_i, \qquad r_i^{min} \le r_i(t) \le r_i^{max} \tag{2}$$

When the battery is discharged, the discharged power is used by other electric appliances. of customer i It is reasonable to assume that the battery cannot discharge more power than the appliances need, i.e., $-r_i(t) \le \sum_{a \in \mathcal{A}_i} q_{i,a}(t)$. Moreover, in order to make sure that there is a certain amount of electric energy in the battery at beginning of the next day, we impose a minimum on the energy level at the end of control horizon: $b(T) \ge \gamma_i B_i$, where $\gamma_i \in (0, 1]$.

The cost of operating the batter is modeled by a function $D_i(r_i)$ that depends on the vector of charged/discharged power $r_i := (r_i(t), t \in \mathcal{T})$. This cost, for example, may correspond to the amortized purchase and maintenance cost of the battery over its lifetime, which depends on how fast/much/often it is charged and discharged. The cost function D_i is assumed to be a convex function of the vector r_i.

III. EQUILIBRIUM AND DISTRIBUTED ALGORITHM

A. Equilibrium

With the battery, at each time t the total power demand of customer i is

$$Q_i(t) \quad := \quad \sum_{a \in \mathcal{A}_i} q_{i,a}(t) + r_i(t) \tag{3}$$

We assume that the utility company is regulated so that its objective is not to maximize its profit through selling electricity, but rather to induce customers' consumption in a way that maximizes the social welfare, total customer utility minus the utility's cost of providing the electricity demanded by all the customers. Hence the utility company aims to solve:

Utility's objective (max welfare):

$$\max_{q,r} \quad \sum_i \left(\sum_{a \in A_i} U_{i,a}(q_{i,a}) - D_i(r_i) \right) - \sum_t C\left(\sum_i Q_i(t) \right) \tag{4}$$

$$\text{s. t.} \quad A^{i,a} q_{i,a} \leq \eta_{i,a}, \quad \forall a, i \tag{5}$$

$$0 \ \leq \ Q_i(t) \ \leq \ Q_i^{\max}, \quad \forall t, i \tag{6}$$

$$r_i \ \in \ \mathcal{R}_i, \quad \forall i \tag{7}$$

where $Q_i(t)$ is defined in (3), the inequality (5) models the various customer appliances (see Section IV for details), the lower inequality of (6) says that customer i's battery cannot provide more power than the total amount consumed by all i's appliances, and the upper inequality of (6) imposes a bound on the total power drawn by customer i. The constraint (7) models the operation of customer i's battery with the feasible set $\mathcal{R}_i$ defined by: for all t, the vectors $r_i \in \mathcal{R}_i$ if and only if

$$0 \ \leq \ b_i(t) \ \leq \ B_i, \quad b_i(T) \geq \gamma_i B_i \tag{8}$$

$$r_i^{min} \ \leq \ r_i(t) \ \leq \ r_i^{max} \tag{9}$$

where $b_i(t)$ is defined in terms of $(r_i(\tau), \tau \leq t)$ in (1).

By assumption, the objective function is concave and the feasible set is convex, and hence an optimal point can in principle be computed centrally by the utility company. This, however, will require the utility company to know all the customer utility and cost functions and all the constraints, which is clearly impractical. The strategy is for the utility company to set prices $p := (p(t), t \in \mathcal{T})$ in order to induce the customers to individually choose the right consumptions and charging schedules (q_i, r_i) in response, as follows.

Given the prices p, we assume that each customer i chooses its own power demand and battery charging schedule $(q_i, r_i) := (q_{i,a}(t), r_i(t), \forall t, \forall a \in \mathcal{A}_i)$ so as to maximize its net benefit, the total utility from operating appliances a at power levels $q_{i,a}$ minus the cost of battery operation and electricity, i.e., each customer i solves:

Customer i's objective (max own benefit):

$$\max_{q_i, r_i} \quad \sum_{a \in \mathcal{A}_i} U_{i,a}(q_{i,a}) - D_i(r_i) - \sum_t p(t) Q_i(t) \tag{10}$$

$$\text{s. t.} \quad (5) - (7)$$

Note that an optimal solution of customers i depends on the prices $p := (p(t), t \in \mathcal{T})$ set by the utility company. We denote it by $(q_i(p), r_i(p)) := (q_{i,a}(t;p), r_i(t;p), \forall t, \forall a \in \mathcal{A}_i)$; similarly, we denote an optimal total power by $Q_i(p) := (Q_i(t;p))$ defined as in (3) but with optimal $q_{i,a}(p)$ and $r_i(p)$.

Definition 1: The prices p and the customer demands $(q, r) := (q_i, r_i, \forall i)$ are in equilibrium if $(q, r) = (q(p), r(p))$, i.e., a solution $(q_i(p), r_i(p))$ to (10) with prices p that is optimal to each customer i is also optimal to the utility company, i.e., maximizes the welfare (4).

The following result follows from the welfare theorem. It implies that setting the price to be the marginal cost of power is optimal.

Theorem 2: There exists an equilibrium p^* and $(q_i^*, r_i^*, \forall i)$. Moreover, $p^*(t) = C'(\sum_i Q_i^*(t)) \geq 0$ for each time t.

Proof: Write the utility company's problem as

$$\max_{(q,r) \in X} \quad \sum_i V_i(q_i, r_i) - \sum_t C\left(\sum_i Q_i(t) \right)$$

$$\text{s. t.} \quad Q_i(t) = \sum_{a \in \mathcal{A}_i} q_{i,a}(t) + r_i(t), \quad \forall i, t$$

where $V_i(q_i, r_i) := \sum_{a \in A_i} U_{i,a}(q_{i,a}) - D_i(r_i)$ and the feasible set X is defined by the constraints (5)–(9). Clearly, an optimal solution (q^*, r^*) exists. Moreover, there exist Lagrange multipliers $p_i^*(t)$, $\forall i, t$, such that (taking derivative with respect to $Q_i(t)$)

$$p_i^*(t) \quad = \quad C'\left(\sum_i Q_i^*(t) \right) \ \geq \ 0$$

Since the right-hand side is independent of i, the utility company can set the prices as $p^*(t) := p_i^*(t) \geq 0$ for all i. One can check that the KKT condition for the utility's problem are identical to the KKT conditions for the collection of customers' problems. Since both the utility's problem and all the customers' problems are convex, the KKT conditions are both necessary and sufficient for optimality. This proves the theorem. ∎

B. Distributed algorithm

Theorem 2 motivates a distributed algorithm where the utility company and the customers jointly compute an equilibrium based on a gradient algorithm, where the utility company sets the prices to be the marginal costs of electricity and each customer solves its own maximization problem in response. The model is that at the beginning of each day, the utility company and (the automated control agents of) the customers iteratively compute the electricity prices $p(t)$, consumptions $q_i(t)$, and charging schedules $r_i(t)$, for each period t of the day, in advance. These decisions are then carried out for that day.

At k-th iteration:

- The utility company collects forecasts of total demands $(Q_i(t), \ \forall t)$ from all customers i over a communication network. It sets the prices to the marginal cost
$$p^k(t) = C'\left(\sum_i Q_i^k(t) \right) \tag{11}$$
and boradcasts $(p^k(t), \forall t)$ to all customers over the communication network.
- Each customer i updates its demand q_i^k and charging schedule r_i^k after receiving the updated p^k, according to
$$\bar{q}_{i,a}^{k+1}(t) = q_{i,a}^k(t) + \gamma \left(\frac{\partial U_{i,a}(q_i^k)}{\partial q_{i,a}^k(t)} - p^k(t) \right) \tag{12}$$
$$\bar{r}_i^{k+1}(t) = r_i^k(t) - \gamma \left(\frac{\partial D_i(r_i^k)}{\partial r_i^k(t)} + p^k(t) \right)$$
$$(q_i^{k+1}, r_i^{k+1}) = \left[\bar{q}_i^{k+1}, \bar{r}_i^{k+1} \right]^{S_i}$$
where $\gamma > 0$ is a constant stepsize, and $[\cdot]^{S_i}$ denotes projection onto the set S_i specified by constraints (5)-(7).

When γ is small enough, the above algorithm converges [12].

IV. Detailed appliance models

In this section, we describe detailed models of electric appliances commonly found in a household. We separate these appliances into four types, each type characterized by a utility function $U_{i,a}(q_{i,a})$ that models how much customer i values the consumption vector $q_{i,a}$, and a set of constraints on the consumption vector $q_{i,a}$. The description in this Section elaborates on the utility functions $U_{i,a}(q_{i,a})$ and the constraint $A^{i,a} q_{i,a} \le \eta_{i,a}$ in the optimization problems defined in Section III.

1) Type 1: The first type includes those appliances such as air conditioner and refrigerator which control the temperature of customer i's environment.

We denote by $\mathcal{A}_{i,1}$ the set of Type 1 appliances for customer i. For each appliance $a \in \mathcal{A}_{i,1}$, $T^{in}_{i,a}(t)$ and $T^{out}_{i,a}(t)$ denote the temperatures at time t inside and outside the place that the appliance is in charge of, and $\mathcal{T}_{i,a}$ denotes the set of timeslots that customer i actually cares about the temperature. For instance, for air conditioner, $T^{in}_{i,a}(t)$ is the temperature inside the house, $T^{out}_{i,a}(t)$ is the temperature outside the house, and $\mathcal{T}_{i,a}$ is the set of timeslots when the resident is at home.

Assume that, at each time $t \in \mathcal{T}_{i,a}$, customer i attains a utility $U_{i,a}(T_{i,a}) := U_{i,a}(T^{in}_{i,a}(t), T^{comf}_{i,a})$ when the temperature is $T^{in}_{i,a}(t)$. The utility function is parameterized by a constant $T^{comf}_{i,a}$ which represents the most comfortable temperature for the customer. We assume that $U_{i,a}(T^{in}_{i,a}(t))$ is a continuously differentiable, concave function of $T^{in}_{i,a}(t)$.

The inside temperature evolves according to the following linear dynamics:

$$T^{in}_{i,a}(t) = T^{in}_{i,a}(t-1) + \alpha(T^{out}_{i,a}(t) - T^{in}_{i,a}(t-1)) + \beta q_{i,a}(t) \tag{13}$$

where α and β are parameters that specify the thermal characteristics of the appliance and the environment in which it operates. The second term in equation (13) models heat transfer. The third term models the thermal efficiency of the system; $\beta > 0$ if appliance a is a heater and $\beta < 0$ if it is a cooler. Here, we define $T^{in}_{i,a}(0)$ as the temperature $T^{in}_{i,a}(T)$ from the previous day. This formulation models the fact that the current temperature depends on the current power draw as well as the temperature in the previous timeslot. Thus the current power consumption has an effect on future temperatures [1], [9], [2]. For each customer i and each appliance $a \in \mathcal{A}_{i,1}$, there is a range of temperature that customer i takes as comfortable, denoted by $[T^{comf,min}_{i,a},\ T^{comf,max}_{i,a}]$. Thus we have the following constraint

$$T^{comf,min}_{i,a} \ \le\ T^{in}_{i,a}(t) \ \le\ T^{comf,max}_{i,a},\ \forall t \in \mathcal{T}_{i,a} \tag{14}$$

We now express the constraints and the argument to the utility functions in terms of the load vector $q_{i,a} := (q_{i,a}(t), \forall t)$. Using equation (13), we can write $T^{in}_{i,a}(t)$ in terms of $(q_{i,a}(\tau), \tau = 1, \dots, t)$:

$$\begin{aligned} T^{in}_{i,a}(t) &= (1-\alpha)^t T^{in}_{i,a}(0) + \sum_{\tau=1}^{t} (1-\alpha)^{t-\tau} \alpha T^{out}_{i,a}(\tau) \\ &\quad + \sum_{\tau=1}^{t} (1-\alpha)^{t-\tau} \beta q_{i,a}(\tau) \end{aligned}$$

Define $T^t_{i,a} := (1-\alpha)^t T^{in}_{i,a}(0) + \sum_{\tau=1}^{t} (1-\alpha)^{t-\tau} \alpha T^{out}_{i,a}(\tau)$,[1] we can further write $T^{in}_{i,a}(t)$ as

$$T^{in}_{i,a}(t) = T^t_{i,a} + \sum_{\tau=1}^{t} (1-\alpha)^{t-\tau} \beta q_{i,a}(\tau) \tag{15}$$

With equation (15), the constraint (14) becomes a linear constraint on the load vector $q_{i,a} := (q_{i,a}(t), \forall t)$: for any $t \in \mathcal{T}_{i,a}$,

$$T^{comf,min}_{i,a} \le T^t_{i,a} + \sum_{\tau=1}^{t} (1-\alpha)^{t-\tau} \beta q_{i,a}(\tau) \le T^{comf,max}_{i,a} \tag{16}$$

The overall utility $U_{i,a}(q_{i,a})$ in the form used in (4) and (10) can then be written in terms of $U_{i,a}(T^{in}_{i,a}(t), T^{comf}_{i,a})$ as[2]

$$U_{i,a}(q_{i,a}) := \sum_{t \in \mathcal{T}_{i,a}} U_{i,a}\left(T^t_{i,a} + \sum_{\tau=1}^{t} (1-\alpha)^{t-\tau} \beta q_{i,a}(\tau), T^{comf}_{i,a} \right) \tag{17}$$

which is a concave function of the vector $q_{i,a}$ since $U_{i,a}(T^{in}_{i,a}(t), T^{comf}_{i,a})$ is concave in $T^{in}_{i,a}(t)$.

In addition, there is a maximum power $q^{max}_{i,a}(t)$ that the appliance can bear at each time, thus we have another constraint on the $q_{i,a}$:

$$0 \le q_{i,a}(t) \le q^{max}_{i,a}(t),\ \forall t$$

2) Type 2: The second category includes the appliances such as PHEV, dish washer, clothes washer. For these appliances, a customer only cares about whether the task is completed before a certain time. This means that the cumulative power consumption by such an appliance must exceed a threshold by the deadline [5], [4], [3].

We denote $\mathcal{A}_{i,2}$ as the set of Type 2 appliances. For each $a \in \mathcal{A}_{i,2}$, $\mathcal{T}_{i,a}$ is the set of times that the appliance can work. For instance, for PHEV, $\mathcal{T}_{i,a}$ is the set of times that the vehicle can be charged. For each customer i and $a \in \mathcal{A}_{i,2}$, we have the following constraints on the load vector $q_{i,a}$:

$$\begin{aligned} q^{min}_{i,a}(t) \ &\le\ q_{i,a}(t) \ \le\ q^{max}_{i,a}(t), \quad \forall t \in \mathcal{T}_{i,a}, \\ q_{i,a}(t) \ &=\ 0, \quad \forall t \in \mathcal{T} \backslash \mathcal{T}_{i,a} \\ \bar{Q}^{min}_{i,a} \ &\le\ \textstyle\sum_{t \in \mathcal{T}_{i,a}} q_{i,a}(t) \ \le\ \bar{Q}^{max}_{i,a} \end{aligned}$$

where $q^{min}_{i,a}(t)$ and $q^{max}_{i,a}(t)$ are the minimum and maximum power load that the appliance can consume at time t, and $\bar{Q}^{min}_{i,a}$ and $\bar{Q}^{max}_{i,a}$ are the minimum and maximum total power draw that the appliance requires. If we set $q^{min}_{i,a}(t) = q^{max}_{i,a}(t) = 0$ for $t \in \mathcal{T} \backslash \mathcal{T}_{i,a}$, we can rewrite these constraints as

$$\begin{aligned} q^{min}_{i,a}(t) \le q_{i,a}(t) \le q^{max}_{i,a}(t),\ \forall t \in \mathcal{T} \\ \bar{Q}^{min}_{i,a} \le \textstyle\sum_{t \in \mathcal{T}_{i,a}} q_{i,a}(t) \le \bar{Q}^{max}_{i,a} \end{aligned} \tag{18}$$

The overall utility that customer i obtains from a Type-2 appliance a depends on the total power consumption by a over

[1] $T^t_{i,a}$ represents the temperature at time t if the appliance a doesn't exist. It is determined by outside temperature and not controlled by the customer.

[2] We abuse notation to use $U_{i,a}$ to denote two different functions; the meaning should be clear from the context.

the whole day. Hence the utility function in the form used in Section III is: $U_{i,a}(q_{i,a}) := U_{i,a}\left(\sum_t q_{i,a}(t)\right)$. We assume that the utility function is a continuously differentiable, concave function of $\sum_t q_{i,a}(t)$.

3) Type 3: The third category includes the appliances such as lighting that must be on for a certain period of time. A customer cares about how much light they can get at each time t. We denote by $\mathcal{A}_{i,3}$ the set of Type-3 appliances and by $\mathcal{T}_{i,a}$ the set of times that the appliance should work. For each customer i and $a \in \mathcal{A}_{i,3}$, we have the following constraints on the load vector $q_{i,a}$:

$$q_{i,a}^{min}(t) \le q_{i,a}(t) \le q_{i,a}^{max}(t), \ \forall t \in \mathcal{T}_{i,a}. \tag{19}$$

At each time $t \in \mathcal{T}_{i,a}$, we assume that customer i attains a utility $U_{i,a}(q_{i,a}(t), t)$ from consuming power $q_{i,a}(t)$ on appliance a. The overall utility is then $U_{i,a}(q_{i,a}) := \sum_t U_{i,a}(q_{i,a}(t), t)$. Again, we assume $U_{i,a}$ is a continuously differentiable, concave function.

4) Type 4:: The fourth category includes the appliances such as TV, video games, and computers that a customer uses for entertainment. For those appliances, the customer cares about two things: how much power they use at each time they want to use the appliance, and how much total power they consume over the entire day.

We denote by $\mathcal{A}_{i,4}$ the set of Type-4 appliances and by $\mathcal{T}_{i,a}$ the set of times that customer i can use the appliance. For instance, for TV, $\mathcal{T}_{i,a}$ is the set of times that the customer is able to watch TV. For each customer i and $a \in \mathcal{A}_{i,4}$, we have the following constraints on the load vector $q_{i,a}$:

$$\begin{array}{l} q_{i,a}^{min}(t) \le q_{i,a}(t) \le q_{i,a}^{max}(t), \ \forall t \in \mathcal{T}_{i,a} \\ \bar{Q}_{i,a}^{min} \le \sum_{t \in \mathcal{T}_{i,a}} q_{i,a}(t) \le \bar{Q}_{i,a}^{max} \end{array} \tag{20}$$

where $q_{i,a}^{min}(t)$ and $q_{i,a}^{max}(t)$ are the minimum and maximum power that the appliance can consume at each time t; $\bar{Q}_{i,a}^{min}$ and $\bar{Q}_{i,a}^{max}$ are the minimum and maximum total power that the customer demands for the appliance. For example, a customer may have a favorite TV program that he wants to watch everyday. With DVR, the customer can watch this program at any time. However the total power demand from TV should at least be able to cover the favorite program.

Assume that customer i attains a utility $U_{i,a}(q_{i,a}(t), t)$ from consuming power $q_{i,a}(t)$ on appliance $a \in \mathcal{A}_{i,4}$ at time t. The time dependent utility function models the fact that the resident would get different benefits from consuming the same amount of power at different times. Take watching the favorite TV program as an example. Though the resident is able to watch it at any time, he may enjoy the program at different levels at different times.

V. Numerical Experiments

In this section, we provide numerical examples to complement the analysis in the previous sections.

A. Simulation setup

We consider a simple system with 8 households in one neighborhood that join in the demand response system. The households are divided into two types evenly. For the households of first type (indexed by $i = 1, 2, 3, 4$), there are residents staying at home for the whole day; for the households of second type (indexed by $i = 5, 6, 7, 8$), there is no person staying at home during the day time (8am-6pm). A day starts at 8am, i.e., $t \in \mathcal{T}$ corresponds to the hour $[7 + t \pmod{24}, 8 + t \pmod{24}]$. Each household is assumed to have 6 appliances: air conditioner, PHEV, clothes washer, lighting, entertainment,[3] and electric battery. The basic parameters of each appliance used in simulation are shown as follows.

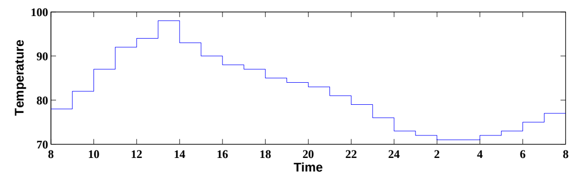

Fig. 1. Outside Temperature over a day

1) Air conditioner: This appliance belongs to Type 1. The outside temperature is shown in Figure 1. It captures a typical summer day in Southern California. For each resident, we assume that the comfortable temperature range is $[70\text{F}, 79\text{F}]$, and the most comfortable temperature is randomly chosen from $[73\text{F}, 77\text{F}]$. The thermal parameters $\alpha = 0.9$ and β is chosen randomly from $[-0.011, -0.008]$. For each household's air conditioner, we assume that $q^{max} = 4000\text{wh}$ and $q^{min} = 0\text{wh}$; and the utility function takes the form of $U_{i,a}(T_i(t)) := c_{i,a} - b_{i,a}(T_{i,a}(t) - T_{i,a}^{com})^2$, where $b_{i,a}$ and $c_{i,a}$ are positive constants. We further assume that the residents will turn off the air conditioner when they go to sleep.[4] The households of the first type care about the inside temperature through the whole day; and the other households care about the inside temperature during the time $\mathcal{T}_{i,a} = \{18, \cdots, 24, 1, \cdots, 7\}$.
2) PHEV: This appliance belongs to Type 2. We assume that the available charging time, $\mathcal{T}_{i,a} = \{18, \cdots, 24, 1, \cdots, 7\}$, is the same for all houses. The storage capacity is chosen randomly from $[5500\text{wh}, 6000\text{wh}]$; and the minimum total charging requirement is chosen randomly from $[4800\text{wh}, 5100\text{wh}]$. The minimum and maximum charging rates are 0w and 2000w. The utility function takes the form of $U_{i,a}(Q) = b_{i,a}Q + c_{i,a}$, where $b_{i,a}$ and $c_{i,a}$ are positive constants.
3) Washer: This appliance belongs to Type 2. For the households of the first type, the available working time is the whole day; for the other households, the available working time is $\mathcal{T}_{i,a} = \{18, \cdots, 24, 1, \cdots, 7\}$. The minimum and maximum total power demands are chosen from $[1400\text{wh}, 1600\text{wh}]$ and $[2000\text{wh}, 2500\text{wh}]$ respectively. The minimum and maximum working rate are 0w and 1500w respectively. The utility function

[3]Here we aggregate different entertainment devices such as TV and PC effectively as one "entertainment" device.

[4]Notice that the outside temperature during 23pm-8am in Southern California is comfortable. It is common that customers turn of air conditioner in the mid-night.

takes the form of $U_{i,a}(Q) = Q + c_{i,a}$, where $c_{i,a}$ is a positive constant.

4) Lighting: This appliance belongs to Type 3. $\mathcal{T}_{i,a} = \{18, \cdots, 23\}$, and the minimum and maximum working power requirements are 200w and 800w respectively. The utility function takes the form of $U_{i,a}(q_{i,a}(t)) = c_{i,a} - (b_{i,a} - \frac{q_{i,a}(t)}{\bar{q}})^{-1.5}$, where $b_{i,a}$ and $c_{i,a}$ are positive constants.

5) Entertainment: This appliance belongs to Type 4. For the households of the first type, $\mathcal{T}_{i,a} = \{12, \cdots, 23\}$, $Q_i^{max} = 3500$wh, and $Q_i^{min} = 1200$wh; for the other households, $\mathcal{T}_{i,a} = \{18, \cdots, 24\}$, $Q_i^{max} = 2000$wh, and $Q_i^{min} = 500$wh. The minimum and maximum working rate are 0w and 400w respectively. The utility function takes the form of $U_{i,a}(q_{i,a}(t)) = c_{i,a} - (b_{i,a} - \frac{q_{i,a}(t)}{\bar{q}})^{-1.5}$, where $b_{i,a}$ and $c_{i,a}$ are positive constants.

6) Battery: The storage capacity is chosen randomly from $[5500\text{wh}, 6500\text{wh}]$ and the maximum charging/discharging rates are both 1800w. We set $\gamma_i = 0.5$, and the cost function takes the following form:

$$\begin{aligned} & D_i\left(r_i\right) \\ = & \Big(\eta_1 \textstyle\sum_{t\in\mathcal{T}} (r_i(t))^2 - \eta_2 \sum_{t=1}^{T-1} r_i(t) r_{i(t+1)} \\ & + \eta_3 \textstyle\sum_{t\in\mathcal{T}} \left(\min(b_i(t) - \delta B_i, 0)\right)^2 + c_{i,b}\Big) \end{aligned}$$

where η_1, η_2, η_3, δ and $c_{i,b}$ are positive constants. The first term captures the damaging effect of fast charging and discharging; the second term penalizes charging/discharging cycles;[5] the third term captures the fact that deep discharge can damage the battery. We set $\delta = 0.2$.[6]

On the supply side, we assume that the electricity cost function is a smooth piecewise quadratic function [13], i.e.,

$$C(Q) = \begin{cases} c_1 Q^2 + b_1 Q + a_1; & 0 \le Q \le Q_1 \\ c_2 Q^2 + b_2 Q + a_2; & Q_1 < Q \le Q_2 \\ \vdots & \vdots \\ c_m Q^2 + b_m Q + a_m; & Q_{m-1} < Q \end{cases}$$

where $c_m > c_{m-1} > \ldots \ge c_1 > 0$.

B. Real-time pricing demand response

Let us first see the performance of our proposed demand response scheme with real-time pricing, without and with battery.

Figure 2 shows the total electricity demand under the real-time pricing demand response scheme without battery; and Figure 3 shows the corresponding electricity allocation for two typical households of different types. We see that different appliances are coordinated indirectly by real-time pricing, so as to flatten the total power demand at different times as much as possible.

[5]If $r(t)$ and $r(t+1)$ have different signs, then there will be a cost. As long as η_2 is smaller than η_1, the cost function is a positive convex function. The second item can also be seen as a correction term to the first term.

[6]We assume that the batteries are lead-acid type batteries rather than NiCd batteries.

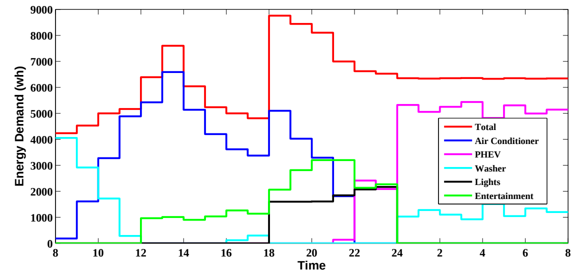

Fig. 2. Total electricity demand under the real-time pricing demand response scheme without battery

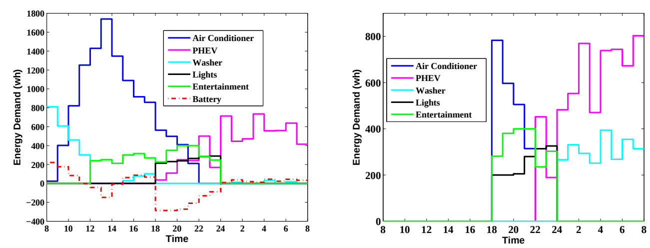

Fig. 3. Electricity demand response for two typical households of different types without battery. The left panel shows the electric energy allocation for the household of the first type. The right panel shows the electric energy allocation for the household of the second type.

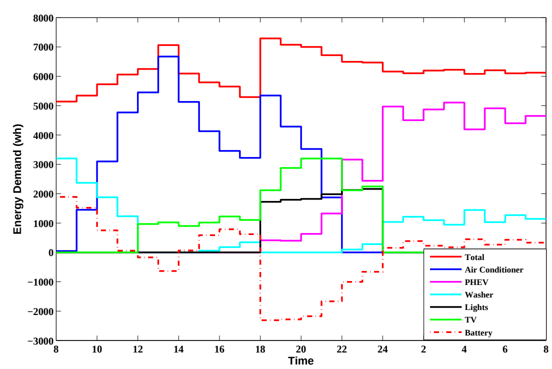

Fig. 4. Total electricity demand under the real-time pricing demand response scheme with battery

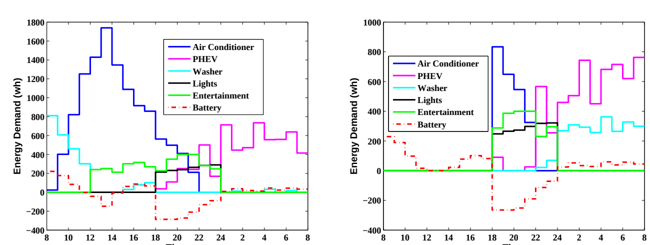

Fig. 5. Electricity demand response for two typical households of different types with battery. The left panel shows the electric energy allocation for the household of the first type. The right panel shows the electric energy allocation for the household of the second type.

Figure 4 shows the total electricity demand under the real-time pricing demand response scheme with battery; and Figure 5 shows the corresponding electricity allocation for two typical households of different types. Those figures show the value of battery for demand response: it does not only reduce the peak load but also helps to further flatten the total power demand at different times.

C. Comparisons among different demand response schemes

In order to evaluate the performance of our proposed demand response scheme, we consider 3 other schemes. In the first scheme the customer is not responsive to any price or cost, just wants to live a comfortable lifestyle; and in the second and third ones, the customer responds to certain flat price.

1) **No demand response:** The customers just allocate their energy usage according to their own preference without paying any attention to the price, i.e., they just optimize their utility without caring about their payment. For example, the customer sets the air conditioner to keep the temperature to the most comfortable level all the time; charges PHEV, washes clothes and watches TV at the favorite times. The electricity demand over a day under this scheme is shown by the blue plot in Figure 6.
2) **Flat price scheme 1:** In this scheme, the customer is charged a flat price p, such that $p = \frac{(1+\Delta)\sum_{t\in\mathcal{T}} C(Q(t),t)}{\sum_{t\in\mathcal{T}} Q(t)}$ with $\{Q(t)\}_{t\in\mathcal{T}}$ the best response to such a price from the customers. To find such a price, we run iterations between the utility company and customers. At each iteration $k = 1, 2, \cdots$, the utility company set the price as $p_k = \frac{(1+\Delta)\sum_{t\in\mathcal{T}} C(Q_k(t),t)}{\sum_{t\in\mathcal{T}} Q_k(t)}$ and then the customers will shape their demand in response to such a flat price. eventually, p_k will converge to a fixed point, which is the flat price we need.[7] The electricity demand over a day under this scheme is shown by the magenta plot in Figure 6.
3) **Flat price scheme 2:** In this scheme we use the information obtained from our proposed real-time pricing demand response scheme to set a flat price p. We collect the price $\{p(t)\}_{t\in\mathcal{T}}$ and total power demand $\{Q(t)\}_{t\in\mathcal{T}}$ information under real time pricing scheme and then set the flat price as $p = \frac{\sum_{t\in\mathcal{T}} p(t)Q(t)}{\sum_{t\in\mathcal{T}} Q(t)}$. The electricity demand over a day under this scheme is shown by the black plot in Figure 6.

Figure 6 also shows the electricity demand response under the real-time pricing scheme with and without battery. We see that the real-time pricing demand response scheme is very effective in shaping the demand: not only the peak load is reduced greatly, but also the variation in power demand decreases greatly; and with the integration of the battery, the peak load and the variation in power demand will be reduced further.

[7]In general, such a price may not exist and the iterative procedure described may not converge.

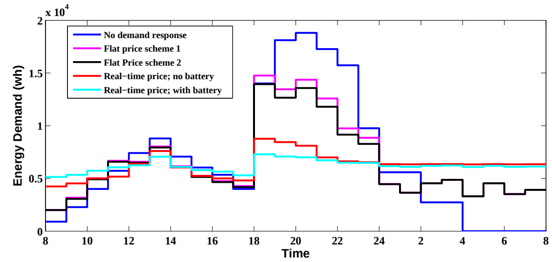

Fig. 6. Electricity demand response under different schemes

TABLE I
DEMAND RESPONSE WITHOUT BATTERY

	No Demand Response	Flat Pricing (Scheme 1)	Flat Pricing (Scheme 2)	Real-time Pricing; no Battery	Real-Time Pricing; with Battery
Load Factor	0.3587	0.4495	0.4577	0.7146	0.8496
Peak Demand	18.8 kwh	14.7 kwh	13 kwh	8.76 kwh	7.29 kwh
Total Demand	162 kwh	158 kwh	153 kwh	150 kwh	148 kwh
Generation Cost	\$64.41	\$45.49	\$41.80	\$32.82	\$31.50
Total Payment	\$137.40[a]	\$ 54.59	\$58.56	\$57.42	\$55.69
Customers' Utility	\$212.41	\$201.72	\$200.14	\$198.82	\$198.82[b]
Customers' Net Utility[c]	\$75.01	\$147.14	\$141.57	\$141.40	\$143.13
Social Welfare	\$148.00	\$156.24	\$158.33	\$166.00	\$167.32

[a]The price at each time slot is set as the real-time marginal generation cost.
[b]When there is a battery, a customer' utility is defined as the benefits the customer gets from electric appliances minus the battery cost.
[c]Customers' net utility is defined as customers' utility minus payment.

Table I summarizes the differences among the three pricing schemes. We see that the real-time pricing scheme can increase the load factor greatly and save a large amount of generation cost without hurting customers' utility; and the integration of the battery can further increase the load factor and reap larger savings in generation cost.

D. Battery with different cost

One of the challenges in the integration of battery is its economic (in)viability because of high battery cost. In order to study the impact of battery cost on demand response, we considers three scenarios with high, mild, and low cost, by choosing different scaling factors (10, 1 and 0.1) for the battery cost in the objective function. Figure 7 shows the electricity demand under the real-time pricing scheme with batteries of different costs. Table II summarizes the differences among those different scenarios. We see that the economic viability of the battery is important, and more economically viable battery will reap more benefits from demand response.

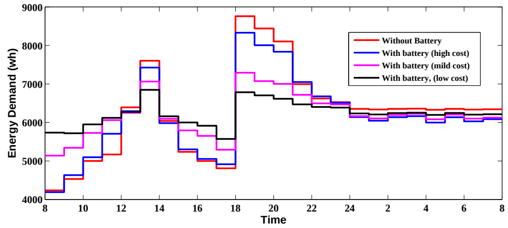

Fig. 7. Electricity demand response with battery at different costs

TABLE II
DEMAND RESPONSE WITH BATTERY

	No Battery	Battery (high-cost)	Battery (mild-cost)	Battery (low-cost)
Load Factor	0.7146	0.7390	0.8496	0.9095
Peak Demand	8.76 kwh	8.33 kwh	7.29 kwh	6.84 kwh
Total Demand	150 kwh	148 kwh	148 kwh	149 kwh
Generation Cost	$32.82	$31.72	$31.50	$31.70
Total Payment	$57.42	$56.35	$55.69	$55.99
Customers' Utility[a]	$198.82	$198.55	$198.82	$199.42
Customers' Net Utility[b]	$141.40	$142.92	$143.13	$143.43
Social Welfare	$166.00	$166.84	$167.32	$167.69

[a] A customer' utility is defined as the benefits the customer gets from electric appliances minus the battery cost.
[b] A customer' utility is defined as the customer's utility minus the payment.

E. Performance scaling with different numbers of households

In order to study the effect of the system size on the performance of our demand response scheme, we simulate systems with the number of customers being $N = 2, 4, 6, \cdots, 24$. Figure 8 summarizes three interesting characteristic factors for the demand response systems with different numbers of households. We see that as the number of households increases, the load factor will first increase till a maximum value and then decrease a bit and finally level off; but the peak load and total demand at each household will decrease and finally level off. This shows that as the number of the households increases, our demand response scheme will reap more benefits but the gain will eventually saturate.

VI. CONCLUSION

We have studied optimal demand response based on utility maximization in power networks. We consider households that operate different appliances including PHEVs and batteries and propose a demand response approach based on utility maximization. Each appliance provides a certain benefit depending on the pattern or volume of power it consumes. Each household wishes to optimally schedule its power consumption so as to maximize its individual net benefit subject to various consumption and power flow constraints. We show that there exist time-varying prices that can align individual optimality with social optimality, i.e., under such prices, when the households selfishly optimize their own benefits, they automatically also maximize the social welfare. The utility company can thus use dynamic pricing to coordinate demand responses to the benefit of the overall system. We propose a distributed algorithm for the utility company and the customers to jointly compute this optimal prices and demand schedules. Finally, we present simulation results that illustrate several interesting properties of the proposed scheme.

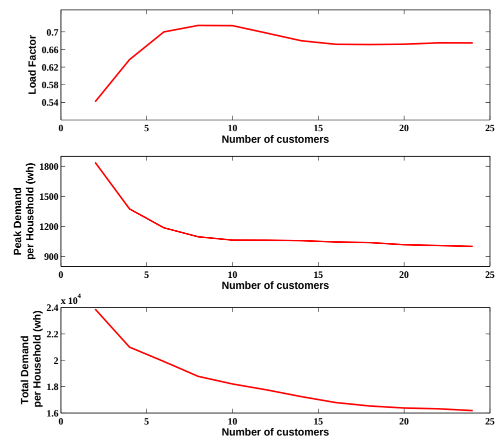

Fig. 8. Electricity demand response without battery for different power networks with different number of customers.

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