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Real-Time Demand Response with Uncertain Renewable Energy in Smart Grid

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***Abstract*—We consider a set of users served by a single load-serving entity (LSE) in the electricity grid. The LSE procures capacity a day ahead. When random renewable energy is realized at delivery time, it actively manages user load through real-time demand response and purchases balancing power on the spot market to meet the aggregate demand. Hence, to maximize the social welfare, decisions must be coordinated over two timescales (a day ahead and in real time), in the presence of supply uncertainty, and computed jointly by the LSE and the users since the necessary information is distributed among them. We formulate the problem as a dynamic program. We propose a distributed heuristic algorithm and prove its optimality when the welfare function is quadratic and the LSE's decisions are strictly positive. Otherwise, we bound the gap between the welfare achieved by the heuristic algorithm and the maximum in certain cases. Simulation results suggest that the performance gap is small. As we scale up the size of a renewable generation plant, both its mean production and its variance will likely increase. We characterize the impact of the mean and variance of renewable energy on the maximum welfare. This paper is a continuation of [2], focusing on time-correlated demand.**

I. Introduction

This paper is the second part of a two-paper series where the first part is [2]. The goal of this paper is to design and evaluate distributed algorithms for optimal energy procurement and demand response in the presence of uncertain renewable supply and time-correlated demand. The overall motivation and the discussion of related literature have been presented in the Introduction section of [2]. In the following, we introduce the problem under study and the contribution of the current paper and earlier papers.

Specifically we consider a set of users that are served by a single load-serving entity (LSE). The LSE may represent a regulated monopoly like most utility companies in the United States today, or a non-profit cooperative that serves a community of end users. Its purpose is (possibly regulated) to promote the overall system welfare. The LSE purchases electricity on the wholesale electricity markets (e.g., day-ahead, real-time balancing, and ancillary services) and sells it on the retail market to end users. Each user, on the other hand, has a set of appliances (electric vehicle, air conditioner, lighting, battery, etc.) which can adapt their demand. The user's energy management system is to decide how much power to consume in each period of a day (i.e., demand response). The LSE is to make energy procurement decisions, including how much capacity it should procure a day ahead and, when the random renewable energy is realized at real time, how much balancing power to purchase on the spot market to meet the aggregate demand. The overall goal is to maximize the social welfare.

Our model captures three important features:

- **Uncertainty.** Part of the electricity supply is from renewable sources such as wind and solar, and thus uncertain.
- **Supply and demand.** LSE's supply decisions and the users' consumption decisions must be jointly optimized.
- **Two timescale.** The LSE must procure capacity on the day-ahead wholesale market while user consumptions should be adapted in real time to mitigate supply uncertainty.

Hence the key is the coordination of day-ahead procurement and real-time demand response over two timescales in the presence of supply uncertainty. Moreover, the optimal decisions must be computed jointly and distributively by the LSE and the users as the necessary information is distributed among them.

In [1], we considered the case without renewable generation. In the absence of uncertainty it becomes unnecessary to adapt user consumptions in real-time and hence supply and consumptions can be optimally scheduled at once instead of over two days. We show that optimal prices exist that coordinate individual users' decisions in a distributed manner, i.e., when users selfishly maximize their own surplus under the optimal prices, their consumption decisions turn out to also maximize the social welfare. We develop a distributed algorithm that jointly schedules the LSE's procurement decisions and the users' consumption decisions for each period in the following day. The algorithm is decentralized where the LSE only knows the aggregate demand but not users' utility functions or consumption constraints, and the users only respond to common prices from the LSE, without coordinating among themselves or knowing the cost functions of the LSE.

With renewable generation, the uncertainty precludes pure scheduling and calls for real-time consumptions decisions that adapt to the realization of the random renewable generation. Moreover, this must be coordinated with procurement

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decisions over two timescales to maximize the expected welfare. Motivated by this, we aim to design distributed algorithms for optimal energy procurement and demand response with renewable supply and understand the impact of uncertainty on the optimal welfare. In [2], we have started this effort with a focus on the case when user demand is not time-correlated. In that case, each time period can be optimized independently. In this paper, we will elaborate on the case when the demand is time-correlated (meaning that the amount of consumption in one period affects the future consumption requirement). Specifically, we will formulate a dynamic program, propose a distributed algorithm, and evaluate its performance both analytically and through simulations. We will also study the impact of renewable energy on the maximum social welfare with time-correlated demand.

II. Model and problem formulation

In this section, we present the mathematical model and problem formulation. The user and supply models have been presented in [2], but they are also included here (Section II-A and II-B) for completeness.

Consider a set $\mathcal{N}$ of N users that are served by a single load-serving entity (LSE). We use a discrete-time model with a finite horizon that models a day. Each day is divided into T periods of equal duration, indexed by $t \in \mathcal{T} = \{1, 2, \cdots, T\}$. The duration of a period can be 5, 15, or 60 mins, corresponding to the time resolution at which energy dispatch or demand response decisions are made.

A. User model

Each user $i \in \mathcal{N}$ operates a set $\mathcal{A}_i$ of appliances such as HVAC (heat, ventilation, air conditioner), refrigerator, and plug-in hybrid electric vehicle. For each appliance $a \in \mathcal{A}_i$ of user i, $q_{ia}(t)$ denotes its energy consumption in period $t \in \mathcal{T}$, and q_{ia} the vector $(q_{ia}(t), \forall t)$ over the whole day. An appliance a is characterized by:

- a utility function $U_{ia}(q_{ia})$ that quantifies the utility user i obtains from using appliance a (where $U_{ia}(q_{ia})$ is continuously differentiable and concave);
- consumption constraints: the vector of power q_{ia} satisfies the linear inequality

$$A_{ia} q_{ia} \le \eta_{ia} \tag{1}$$

where A_{ia} is a $K_{ia} \times T$ matrix and η_{ia} is a K_{ia}-vector. This model is quite flexible. In [1], by unifying several models in the literature, we specified the utility functions and linear consumption constraints of different types of appliances (such as those mentioned above). A concrete example will be given later in Eq. (6), (7), and (8).

B. Supply model

We now describe a simple model of the electricity markets. The LSE procures power for delivery in each period t, in two steps. First, one day in advance, it procures "day-ahead" capacities $P_d(t)$ for each period t of the day under consideration, and pays for the capacity costs $c_d(P_d(t); t)$. The renewable power in each period t is a nonnegative random variable $P_r(t)$ and it costs $c_r(P_r(t); t)$. It is desirable to use as much renewable power as possible; for notational simplicity only, we assume $c_r(P; t) \equiv 0$ for all $P \ge 0$ and all t. Then at time t^- (real time), the random variable $P_r(t)$ is realized and used to satisfy demand. The LSE satisfies any excess demand by some or all of the day-ahead capacity $P_d(t)$ procured in advance and/or by purchasing balancing power on the real-time market. Let $P_o(t)$ denote the amount of the day-ahead power that the LSE actually uses and $c_o(P_o(t); t)$ its "operation cost" (in addition to the capacity cost c_d). Let $P_b(t)$ be the real-time balancing power and $c_b(P_b(t); t)$ its cost. We assume that, for each t, $c_d(\cdot; t)$, $c_o(\cdot; t)$ and $c_b(\cdot; t)$ are increasing, convex, and continuously differentiable with $c_d(0; t) = c_o(0; t) = c_b(0; t) = 0$.

These real-time decisions $(P_o(t), P_b(t))$ are made by the LSE so as to minimize its total cost, as follows. Given the demand vector $q(t) := (q_{ia}(t), \forall a \in \mathcal{A}_i, \forall i)$, let $Q(t) := \sum_{i,a} q_{ia}(t)$ be the total demand and $\Delta(Q(t)) := Q(t) - P_r(t)$ the excess demand, in excess of the renewable generation $P_r(t)$. Note that $\Delta(Q(t))$ is a random variable in and before period $t-1$, but its realization is known to the LSE at time t^-. Given excess demand $\Delta(Q(t))$ and day-ahead capacity $P_d(t)$, the LSE chooses $(P_o(t), P_b(t))$ that minimizes its total real-time cost, i.e., it chooses $(P_o^*(t), P_b^*(t))$ that solves the problem:

$$\begin{aligned} & c_s(\Delta(Q(t)), P_d(t); t) \\ := \quad & \min_{P_o(t), P_b(t)} \{ \, c_o(P_o(t); t) + c_b(P_b(t); t) \mid P_b(t) \ge 0, \\ & P_o(t) + P_b(t) \ge \Delta(Q(t)), \;\; P_d(t) \ge P_o(t) \ge 0 \}. \end{aligned} \tag{2}$$

Clearly $P_o^*(t) + P_b^*(t) = \Delta(Q(t))$ unless $\Delta(Q(t)) < 0$. The total cost is

$$\begin{aligned} & c(Q(t), P_d(t); P_r(t), t) \\ := \quad & c_d(P_d(t); t) + c_s(\Delta(Q(t)), P_d(t); t). \end{aligned} \tag{3}$$

with $\Delta(Q(t)) = Q(t) - P_r(t)$.

Example: supply cost

Suppose $c_b'(0) > c_o'(P), \forall P \ge 0$, i.e., the marginal cost of balancing power is strictly higher than the marginal operation cost of day-ahead power, the LSE will use the balancing power only after the day-ahead power is exhausted, i.e., $P_b(t) > 0$ only if $\Delta(Q(t)) > P_d(t)$. The solution $c_s(\Delta(Q(t)), P_d(t); t)$ of (2) in this case is particularly simple and (3) can be written explicitly in terms of c_b, c_o, c_b:

$$\begin{aligned} & c(Q(t), P_d(t); P_r(t), t) = c_d(P_d(t); t) + \\ & c_o\left([\Delta(Q(t))]_0^{P_d(t)}; t\right) + c_b\left([\Delta(Q(t)) - P_d(t)]_+; t\right). \end{aligned} \tag{4}$$

i.e., the total cost consists of the capacity cost c_d, the energy cost c_o of day-ahead power, and the cost c_b of the real-time balancing power.

C. Problem formulation: welfare maximization

Recall that $q := (q(t),\ t \in \mathcal{T})$ and $Q(t) := \sum_{i,a} q_{ia}(t)$. The social welfare is the standard user utility minus supply cost:

$$W(q, P_d; P_r) := \sum_{i,a} U_{ia}(q_{ia}) - \sum_{t=1}^{T} c(Q(t), P_d(t); P_r(t), t) \tag{5}$$

where $P_d := (P_d(t),\ t \in \mathcal{T})$ and $P_r := (P_r(t),\ t \in \mathcal{T})$. Recall that the LSE's objective is not to maximize its profit through selling electricity, but rather to maximize the expected social welfare. Given the day-ahead decision P_d, the real-time procurement $(P_o(t), P_b(t))$ is determined by the simple optimization (2). This is most transparent in (4) for the special case: the optimal decision is to use day-ahead power $P_o^*(t)$ to satisfy any excess demand $\Delta(Q(t))$ up to $P_d(t)$, and then purchase real-time balancing power $P_b^*(t) = [\Delta(Q(t)) - P_d(t)]_+$ if necessary. Hence the maximization of (5) reduces to optimizing over day-ahead procurement P_d and real-time consumption q in the presence of random renewable generation $P_r(t)$. It is critical that, in the presence of uncertainty, $q(t)$ should be decided after $P_r(t)$ have been realized at times t^- (i.e., real-time demand response). P_d however must be decided a day ahead before $P_r(t)$ are realized. Therefore, the day-ahead procurement and the real-time demand response must be coordinated over two timescales to maximize the expected welfare. The optimal policy is the solution a dynamic programming problem, as detailed below.

Social welfare maximization as a $(T+1)$-stage dynamic program

Consider a time horizon of $T+1$ slots[1]: slot $0, 1, 2, \ldots, T$. Slot 0 corresponds to "day-ahead". At this time, the LSE needs to decide $P_d(t), t = 1, \ldots, T$, based on the distribution of $\{P_r(t), t = 1, \ldots, T\}$. Slot $1, \ldots, T$ are the actual time slots in the day. In slot t, the users decide their consumption.

To simplify the notation we use without loss of generality a simplified user model where each user i has a single appliance (e.g., an electric vehicle), so we drop the subscript a. Also assume that the utility functions are additive in time,

$$U_i(q_i) = \sum_t U_i(q_i(t); t), \tag{6}$$

and the consumption constraints are

$$\underline{q}_i(t) \le q_i(t) \le \overline{q}_i(t), \quad \forall i \tag{7}$$

$$\textstyle\sum_{t=1}^{T} q_i(t) \ge \overline{Q}_i, \quad \forall i \tag{8}$$

That is, the consumption in each period is in some range, and the total consumption must exceed $\overline{Q}_{ia}$. If the appliance cannot use electricity in some period t', then we can define $\underline{q}_{ia}(t') = \overline{q}_{ia}(t') = 0$. Clearly, constraints (7) and (8) are linear and can be written in the form of (1).

For convenience, we further assume that $c_b'(0) > c_o'(P), \forall P \ge 0$. As explained in Section II-B, the total cost c in the welfare function (5) is then given by (4). We now describe the dynamic program.

[1]Throughout the paper, "slot" means "period".

The input in slot 0 is $v(0) = P_d = (P_d(\tau), \tau = 1, \ldots, T) \in \Re_+^T$. For $t = 1, \ldots, T$, the inputs $v(t) = q(t) \in [\underline{q}(t), \overline{q}(t)]$ where $\underline{q}(t) := (\underline{q}_i(t), \forall i)$ and $\overline{q}(t) := (\overline{q}_i(t), \forall i)$. The system state $x(t) := (x^1(t), x^2(t), x^3(t))$ consists of three components of appropriate dimensions such that

$$x(t) \;\; = \;\; (P_d, x^2(t), P_r(t)), \qquad t = 1, \ldots, T$$

where $x^2(t)$ is determined by the consumption constraints. The constraint (8) motivates a state variable $x_i^2(t)$ that tracks remaining demand for user i at the beginning of each period t: define $x_i^2(0) = 0$, $x_i^2(1) = \overline{Q}_i$, and for each $t = 1, \ldots, T$, $x_i^2(t+1) = x_i^2(t) - v_i(t)$ where $v_i(t) = q_i(t)$. To enforce that $x^2(T+1) \le 0_N$, we define a terminal cost $c_{T+1}(x(T+1)) = 0$ if $x^2(T+1) \le 0_N$ and $c_{T+1}(x(T+1)) = \infty$ otherwise, where 0_n is the n-dimensional zero vector.

Let the initial state be $x(0) = 0_{T+N+1}$. Denote $\overline{Q} := (\overline{Q}_i, \forall i)$. The system dynamics is then linear time-varying:

$$x(1) = x(0) + \begin{pmatrix} I_T \\ 0_{(N+1)\times T} \end{pmatrix} v(0) + \begin{pmatrix} 0_T \\ \overline{Q} \\ P_r(1) \end{pmatrix} \tag{9}$$

$$\begin{aligned} x(t+1) \;\; &= \;\; \begin{pmatrix} I_{T+N} & 0_{T+N} \\ 0_{T+N} & 0 \end{pmatrix} x(t) - \begin{pmatrix} 0_{T\times N} \\ I_N \\ 0 \end{pmatrix} v(t) \\ &\quad + \begin{pmatrix} 0_{T+N} \\ 1 \end{pmatrix} P_r(t+1), \ \forall 1 \le t \le T \end{aligned} \tag{10}$$

where I_n is the $n \times n$ identify matrix, $0_{m\times n}$ is the $m \times n$ zero matrix, and $P_r(T+1) := 0$.

The welfare in each period, under input sequence v, is (using (4))

$$\begin{aligned} W_0^v(x(0), v(0)) \;\; &:= \;\; -\sum_{\tau=1}^{T} c_d(P_d(\tau); \tau) \\ &= \;\; -\sum_{\tau=1}^{T} c_d([v(0)]_\tau; \tau) \end{aligned} \tag{11}$$

and for $t = 1, \ldots, T$,

$$\begin{aligned} & W_t^v(x(t), v(t)) \\ := \;\; & \sum_i U_i(q_i(t); t) - c_o\left([Q(t) - P_r(t)]_0^{P_d(t)}; t \right) \\ & -c_b\left([Q(t) - P_r(t) - P_d(t)]_+; t \right) \\ = \;\; & \sum_i U_i(v_i(t); t) - c_o\left(\left[\mathbf{1}v(t) - x^3(t) \right]_0^{[x^1(t)]_t}; t \right) \\ & -c_b\left(\left[\mathbf{1}v(t) - x^3(t) - [x^1(t)]_t \right]_+; t \right) \end{aligned} \tag{12}$$

where $\mathbf{1}$ is the (row) vector of 1's.

Let $\phi := \{\phi_0 : \Re^{T+N+1} \to \Re_+^T,\ \ \phi_t : \Re^{T+N+1} \to [\underline{q}(t), \overline{q}(t)],\ \ t = 1, \ldots, T\}$ be the control policy so that $v(t) = \phi_t(x(t)),\ 0 \le t \le T$. We assume that the joint distribution of $\{P_r(\tau), \tau = 1, \ldots, T\}$ is known. The objective is to choose a control policy ϕ that maximizes the expected welfare:

$$\max_{\phi} \quad E\ \left(\sum_{t=0}^{T} W_t^v(x(t), v(t)) - c_{T+1}\left(x(T+1)\right)\right) \quad (13)$$

where the state $x(t)$ and the input $v(t)$ are obtained under policy ϕ even though this is not explicit in the notation.

Reference [1] also gives a dynamic-programming formulation with more general utility functions and consumption constraints.

III. Online algorithm

A. Algorithm

We propose the following algorithm to solve (13). The algorithm provides an exact solution to the dynamic programming in certain cases and is a heuristic algorithm otherwise.

Algorithm 1: Real-time demand response with uncertain renewable energy

We use P_d^* to denote the choice of day-ahead energy and $q^*(t) := (q_i^*(t), \forall i), t = 1, 2, \ldots, T$ to denote the choice of demand in slot t under Algorithm 1.

1) One day ahead, determine the day-ahead energy $P_d^*(t), t = 1, 2, \ldots, T$ as follows. Solve the (deterministic) optimization problem

$$\begin{array}{ll} \max_{q, P_d \geq 0} & W(q, P_d; \bar{P}_r) \\ s.t. & (7), (8) \end{array} \quad (14)$$

where W is the welfare function defined in (5) (but with the subscript a dropped), $q = (q_i(t), \forall i, t)$, and $\bar{P}_r = E(P_r)$ with $P_r = (P_r(t), \forall t \in \mathcal{T})$. In other words, we maximize the social welfare assuming that the renewable energy is fixed at $\bar{P}_r$. Let the solution of (14) be $(\tilde{q}, \tilde{P}_d)$. Use $\tilde{P}_d$ as the day-ahead energy, i.e., let $P_d^* = \tilde{P}_d$.
2) Let $t = 1$.
3) In period t, determine the consumption of each user in this period as follows. Note that at this time $\{P_r(\tau), 0 \leq \tau \leq t\}$ have been observed by the LSE. So, the conditional distribution of $\{P_r(\tau), \tau > t\}$ is known. Denote

$$\bar{P}_r^t := E(P_r | P_r(\tau), \forall \tau \leq t).$$

Solve the following problem:

$$\begin{array}{ll} \max_q & W(q, P_d^*; \bar{P}_r^t) \\ s.t. & (7), (8) \\ & q_i(\tau) = q_i^*(\tau), \forall \tau < t, \forall i \end{array} \quad (15)$$

where $q_i^*(\tau), \tau < t$ is the consumption of user i in slot $\tau < t$ that is already decided in the earlier slot τ. That is, we maximize the social welfare, given the decisions already made before slot t (i.e., P_d^* and $q_i^*(\tau), \forall \tau < t, \forall i$) and the current $P_r(t)$, and assuming that future renewable energy is fixed at $\bar{P}_r^t(\tau), \tau > t$.
Let the solution of (15) be $\tilde{q}^t$. Use $\tilde{q}^t(t)$ as the consumption in slot t, i.e., let $q^*(t) = \tilde{q}^t(t)$. Finally, choose $P_o^*(t) = [\sum_i q_i^*(t) - P_r(t)]_0^{P_d^*(t)}$, and the real-time energy as $P_b^*(t) = [\sum_i q_i^*(t) - P_r(t) - P_d^*(t)]_+$.
4) If $t < T$, increment t and repeat step 3.

Distributed implementation of Algorithm 1:

Step 1 and 3 of Algorithm 1 needs to solve deterministic optimization problems (14) and (15). This can be done using a distributed gradient projection method, without requiring the LSE and users exchange their private information (cost functions and utility functions). The method is also used in Algorithm 1 in [1] for the case without random renewable energy. For completeness, we write down the algorithm for solving (14). The algorithm for solving (15) is similar.

For each iteration $k = 1, 2, \ldots$, after initialization:

1) The LSE collects demand forecasts, denoted by $(q_i^k(t), \ \forall t)$, from all users i over a communication network. It sets the prices to the marginal costs $\pi^k(t) := \frac{\partial c}{\partial Q(t)}(Q(t), P_d^k(t); \bar{P}_r(t), t)\ |_{Q(t)=\sum_i q_i^k(t)}$ and broadcasts $(\pi^k(t), \forall t)$ to all users. Also, the LSE updates its plan of day-ahead energy according to

$$P_d^{k+1}(t) = [P_d^k(t) - \gamma_k \frac{\partial c}{\partial P_d^k(t)}(\sum_i q_i^k(t), P_d^k(t); \bar{P}_r(t), t)]_0^{P_{max}}, \forall t$$

where $P_{max} := \sum_i \overline{q}_i(t)$, and $\gamma_k > 0$ is the step size.
2) After receiving π^k, each user i computes

$$\hat{q}_i^{k+1}(t) \quad := \quad q_i^k(t) + \gamma_k \left(\frac{\partial U_i\left(q_i^k(t); t\right)}{\partial q_i^k(t)} - \pi^k(t)\right), \quad \forall t \in \mathcal{T}$$

and updates its demand forecasts q_i^{k+1} according to

$$q_i^{k+1} = [\hat{q}_i^{k+1}]_{\mathcal{Q}_i}$$

where $\hat{q}_i^{k+1} = (\hat{q}_i^{k+1}(t), \forall t \in \mathcal{T})$ and $[\cdot]_{\mathcal{Q}_i}$ denotes the projection onto the feasible set $\mathcal{Q}_i$ specified by constraints (7)–(8).
3) Increment iteration index to $k+1$ and go to Step 1.

Since this is a gradient projection algorithm, it is not difficult to show that if the step size $\gamma_k = 1/k, \forall k$, the algorithm converges to the optimal solution of (14), assuming that U_i is strictly concave and c_d is strictly convex.

B. Performance analysis

We first show that under certain conditions, Algorithm 1 provides the optimal solution for our dynamic programming problem.

Proposition 1: Algorithm 1 provides the optimal solution for the dynamic programming problem if the following conditions hold:

(i) The cost functions are quadratic: $c_d(P; t) = a_t P^2 + b_t P$ and $c_b(P; t) = a_t' P^2 + b_t' P$ where $a_t, a_t' > 0$. For simplicity assume that $c_o(P; t) = 0$. The utility function is also quadratic $U_i(q_i(t); t) = \hat{a}_t q_i^2(t) + \hat{b}_t q_i(t) + \hat{c}_t$ where $\hat{a}_t < 0$.

(ii) Constraint (8) is replaced by an equality constraint $\sum_{t=1}^{T} q_i(t) = \overline{Q}_i, \forall i$. (Accordingly, problems (14) and (15) need to use the constraint as well.)

(iii) For any realization of P_r, when solving problem (14) in Algorithm 1, constraint (7) and $P_d \geq 0$ are not active, and $\tilde{P}_b^0(\tau) := [\sum_i \tilde{q}_i(\tau) - \bar{P}_r(\tau) - P_d^*(\tau)]_+ > 0, \forall 1 \leq \tau \leq T$. When solving problem (15), the constraints (7) are never active, and $\tilde{P}_b^t(\tau) := [\sum_i \tilde{q}_i^t(\tau) - \bar{P}_r^t(\tau) - P_d^*(\tau)]_+ > 0, \forall 1 \leq t, \tau \leq T$.

Proof: The proof is given in Appendix A, and is related to linear quadratic stochastic control. ■

If the conditions in Proposition 1 are not satisfied, we can still use Algorithm 1 as a heuristic algorithm. The simulations in Section V show good performance of Algorithm 1.

In this case, it is also interesting to analytically quantify the performance gap between Algorithm 1 and the optimal dynamic-program solution. This task, however, is not easy in general. For notation convenience, in this paper we present the performance gap in a simple setting. The techniques can also be applied to bound the performance gap in other settings.

Proposition 2: Assume that $c_d(P;t) = c_b(P;t) = \frac{1}{2}P^2$ and $c_o(P;t) = 0$ for all t and $P \geq 0$. Assume that there is only one user with index 1. In the consumption constraint (7) with $i = 1$, assume that $\underline{q}_1(t) = 0$ and $\overline{q}_1(t) = +\infty$. That is, the constraint is $q_1(t) \geq 0$. Also assume that the utility function $U_1(q_1(t); t) = 0, \forall t, q_1(t)$. Therefore, our problem is to supply at least $\overline{Q}_1$ amount of energy to user 1 with the minimal expected cost. Finally, assume that $P_r(t)$ are independent across t, and that $\sum_t E(P_r(t)) < \overline{Q}_1$.

Let J and J^* be the expected welfare achieved by Algorithm 1 and the optimal dynamic-program solution, respectively. Then the following bound holds:

$$J^* - \sum_{t=1}^{T} \frac{1}{T - t + 1} \sigma^2(t) \leq J \leq J^*,$$

where $\sigma^2(t)$ is the variance of $P_r(t)$. (For example, if $\sigma^2(t) = \sigma^2, \forall t$, then the performance gap is bounded by $\log(T) \cdot \sigma^2$.)

Proof: See Appendix B. ■

C. Extension of Algorithm 1 to the general user model

In our general user model in section II-A, a user can have multiple appliances, and for each appliance, the utility function may not be separable in t, and the consumption constraints can be different from (7) and (8).

However, it is not difficult to extend Algorithm 1 to the more general case. Specifically, we can simply modify problem (14) and (15), by plugging in the general welfare function (5) and replacing the constraints (7) and (8) with (1). The extended algorithm admits similar distributed implementation. On the other hand, its performance analysis (for example, bounding the optimality gap) may be more difficult.

IV. Impact of Renewable Energy on the Optimal Cost

An important element in our model is the uncertain renewable energy. In the future, the penetration of renewable energy and its impact are expected to increase. In this section, we investigate how the statistics of the renewable energy affects

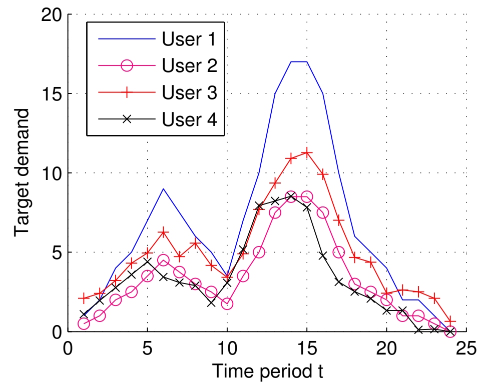


Fig. 1: Target demand profiles of the users

the achievable social welfare in our model. For simplicity, assume that the amounts of renewable energy in different slots are independent. Assume that the renewable energy is parametrized by $a \geq 0$ and $b \geq 0$ as follows.

$$P_r(t; a, b) = a \cdot \mu_r(t) + b \cdot V_r(t) \geq 0, \forall t$$

where $\mu_r(t)$ is a constant, and $V_r(t)$ is a zero-mean random variable. So, a and b indicate the mean and variance, respectively, of the renewable energy. In particular, $E[P_r(t; a, b)] = a \cdot \mu_r(t)$, and $var[P_r(t; a, b)] = b^2 E[V_r^2(t)]$.

Let $J^*(a, b)$ be the maximal expected welfare (resulting from the dynamic program (13)) when the renewable energy is $P_r(t; a, b)$ for all t. Then we have the following results.

Proposition 3: (i) If $b \geq 0$ is fixed, $J^*(a, b)$ is non-decreasing with $a \geq 0$.

(ii) If $a \geq 0$ is fixed, $J^*(a, b)$ is non-increasing with $b \geq 0$.

(iii) Assume that $\mu_r(t) + V_r(t) \geq 0, \forall t$, so that $P_r(t; s, s) = s \cdot [\mu_r(t) + V_r(t)] \geq 0, \forall t, \forall s \geq 0$. Then $J^*(s, s)$ is non-decreasing with $s \geq 0$.

Remark: In other words, the maximal expected welfare increases with the mean, decreases with the variance, and increases with the scale of P_r.

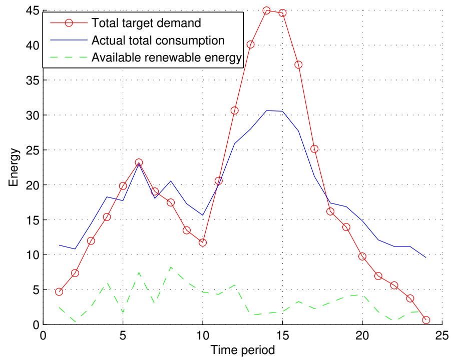
Proof: See Appendix C. ■

V. Numerical Results

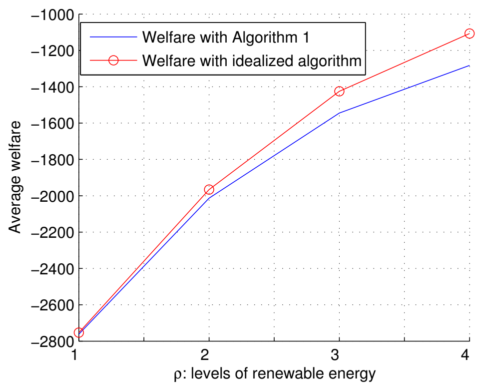
First we illustrate the characteristics of the solution given by Algorithm 1. The setup is as follows. Let $T = 24$, representing 24 hours. The first time period is 8-9am, the second is 9-10am, and so on. The utility function of user i is $U_i(q_i) = \sum_{t=1}^{T} U_i(q_i(t); t) = -\sum_{t=1}^{T} [q_i(t) - y_i(t)]^2$ where $y_i(t)$ is user i's target consumption in slot t. That is, the further his actual demand profile $\{q_i(t)\}$ deviates from the target, the less is his utility. Fig. 1 shows the target demand profiles of $N = 4$ users in our simulation. The unit of energy is kWh.

We impose the constraint that $\sum_t q_i(t) \geq \sum_i y_i(t)$. That is, user i can shift his demand from one time period to another, but his total consumption $\sum_t q_i(t)$ must be at least $\sum_i y_i(t)$.

Assume that $P_r(t)$ is uniformly distributed between 0 and $2\bar{P}_r(t) > 0$, so that its mean is $E(P_r(t)) = \bar{P}_r(t)$. Also,



(a) Consumption



(b) Comparison with idealized algorithm

Fig. 2: Numerical evaluations of Algorithm 1

$P_r(t)$'s are independent across t. The values of $(\bar{P}_r(t), \forall t)$ are (2, 3, 4, 5, 5, 6, 6, 7, 6, 5, 4, 3, 2, 2, 3, 4, 4, 4, 4, 3, 3, 2, 2, 2). For each time period, assume that the cost functions are $c_d(P) = (P^2 + P)/2$, $c_o(P) = P/2$, and $c_b(P) = P^2/2 + 5P$.

We run Algorithm 1 and obtain the consumption of each user. Under one realization of $\{P_r(t)\}$, Fig. 2 (a) shows the total target load (of the 4 users), the total actual consumption, and the realization of $\{P_r(t)\}$. We observe that

- The consumption tends to follow the trend of target demand profile;
- but through demand response, the users tend to opportunistically use the available renewable energy, and at the same time flatten their consumption.

Next, to understand Algorithm 1's performance as compared to the dynamic-programming solution, we compare Algorithm 1 with an idealized algorithm which assumes that the realization of $(P_r(t), \forall t \in \mathcal{T})$ is known one day in advance. The idealized algorithm simply chooses P_d and q to maximize the welfare given the realization of P_r. So, the idealized algorithm achieves even higher welfare than the dynamic-programming solution, and our comparison is therefore conservative. We consider different penetration levels of the renewable energy. Specifically, we let the mean of renewable energy be $\rho \bar{P}_r(t), \forall t$. For each $\rho \in \{1, 2, 3, 4\}$, we repeat the simulation 30 times (each time with a realization of the renewable energy) and compute the average welfare (as an approximation of the expected welfare). The average welfare achieved by Algorithm 1 and the idealized algorithm is shown in Fig. 2 (b). Note that the average renewable energy is 15.36% of the total target demand when $\rho = 1$, and is 61.45% when $\rho = 4$. Algorithm 1 achieves reasonable performance even when the penetration of renewable energy is high.

VI. Conclusion

In this paper, we have studied multi-period energy procurement and demand responses in the presence of uncertain supply of renewable energy. We have provided a decentralized algorithm with two-way communications for the load-serving entity and the users, aiming to efficiently use the grid and maximize social welfare. We have studied the performance of the algorithm through both analysis and simulations. We have provided insight on the effect of clean, but random renewable energy on the social welfare.

Here, we have focused on one type of utility functions and consumption constraints. In the future, we will incorporate other types of appliances as well, such as those modeled in [1]. Note that Algorithm 1 can be easily extended to that case. The challenges lie in understanding its performance in more general settings, and possibly designing more efficient algorithms. Also, we are interested in considering the case with distributed renewable generations on the user side, which will become more common in the future, and investigate how that changes the structure of optimal energy procurement and demand response strategies.

Appendix A
Proof of Proposition 1

Proof: Consider a dynamic-programming problem, called DP2, modified from our DP considered in this proposition. Specifically, we remove the constraints $P_d \geq 0$ and (7), use the constraint $\sum_{t=1}^{T} q_i(t) = \overline{Q}_i$ as assumed, and change the cost $c_b\left([Q(t) - P_r(t) - P_d(t)]_+ ; t\right)$ to $c_b\left(Q(t) - P_r(t) - P_d(t); t\right)$. Accordingly, we expand the ranges of $c_d(P;t)$ and $c_b(P;t)$ to $P \in (-\infty, \infty)$. Since we have assumed $c_o(\cdot;t) = 0$, DP2 is a linear quadratic stochastic control problem.

We first show that Algorithm 1 provides the optimal solution of DP2. Note that this result is standard if $\{P_r(t)\}$ are independent of each other, using the "certainty equivalence principle". Since in general $\{P_r(t)\}$ are correlated, we need the following proof to show the result.

We expand the state of renewable energy at time $t-$ into $x^3(t) = E(P_r|P_r(\tau), \tau \leq t)$ where $P_r = (P_r(t), t = 1, 2, \ldots, T)$. Note that the first t elements of $x^3(t)$ are the observed renewable energy in the first t slots, and the other elements are conditional expectations of future renewable energy. Denote $\bar{P}_r^t := E(P_r|P_r(\tau), \tau \leq t) = x^3(t)$.

The overall state evolves linearly as follows:

$$x(t+1) = C_t \cdot x(t) + D_t \cdot v(t) + F_t \cdot \bar{P}_r^{t+1}$$

with proper matrices C_t, D_t and F_t.

Since the utility functions $u_i(\cdot;t)$ and cost functions $c_d(\cdot;t), c_b(\cdot;t)$ are quadratic by assumption (and $c_o(\cdot;t) = 0$), the optimal welfare in the last stage is a quadratic form involving $x(T)$:

$$J_T(x(T)) = x(T)^{'} P_T \cdot x(T) + \bar{A}_T x(T) + K_T. \qquad (16)$$

where $P_T, \bar{A}_T$ are constant matrices, and K_T is a constant scalar.

So,

$$\begin{aligned} & J_{T-1}(x(T-1)) \\ = \quad & \max_v \{(A_{T-1} \cdot x(T-1) + G_{T-1}v)^{'} P_{T-1}(A_{T-1} \cdot \\ & x(T-1) + G_{T-1}v) + [\bar{A}_{T-1}x(T-1) + \bar{G}_{T-1}v] \\ & + K_{T-1} + E_{T-1}[J_T(C_{T-1} \cdot x(T-1) + \\ & D_{T-1} \cdot v + F_{T-1} \cdot \bar{P}_r^T + h_{T-1})]\} \qquad (17) \end{aligned}$$

where $E_{T-1}(\cdot)$ denotes the conditional expectation given $P_r(\tau), \tau \leq T-1$, and $A_{T-1}, G_{T-1}, \ldots$ are all constant matrices or scalars.

Note that $E_{T-1}(\bar{P}_r^T) = \bar{P}_r^{T-1}$. So we can write $\bar{P}_r^T = \bar{P}_r^{T-1} + w_T$ where $E_{T-1}(w_T) = 0$. So

$$\begin{aligned} & E_{T-1}[J_T(C_{T-1} \cdot x(T-1) + D_{T-1} \cdot v \\ & \quad + F_{T-1} \cdot \bar{P}_r^T + h_{T-1})] \\ = \quad & E_{T-1}[J_T(C_{T-1} \cdot x(T-1) + D_{T-1} \cdot v \\ & \quad + F_{T-1} \cdot (\bar{P}_r^{T-1} + w_T) + h_{T-1})]. \end{aligned}$$

Using (16), we have

$$\begin{aligned} & E_{T-1}[J_T(C_{T-1} \cdot x(T-1) + D_{T-1} \cdot v \\ & \quad + F_{T-1} \cdot (\bar{P}_r^{T-1} + w_T) + h_{T-1})] \\ = \quad & (\tilde{A}_{T-1} \cdot x(T-1) + \tilde{G}_{T-1}v)^{'} P_T(\tilde{A}_{T-1} \cdot x(T-1) + \\ & \tilde{G}_{T-1}v) + [\tilde{\bar{A}}_{T-1}x(T-1) + \tilde{\bar{G}}_{T-1}v] \\ & + \tilde{K}_T + E_{T+1}[w_T^{'} \tilde{P}_T w_T]. \end{aligned}$$

Note that the RHS does not have a $\bar{P}_r^{T-1}$ term because it has been absorbed in $\tilde{A}_{T-1} \cdot x(T-1)$. Also, note that the last term $E_{T+1}[w_T^{'} \tilde{P}_T w_T]$ is a constant.

So, the solution to the RHS of (17) is a linear function of $x(T-1)$. As a result, $J_{T-1}(x(T-1))$ is quadratic in $x(T-1)$. Continuing the backwards induction, we know that $J_t(x(t)), t < T-1$ also has quadratic form.

In Algorithm 1, for slot t, we choose $v(t)$ assuming that future renewable energy is deterministically $\bar{P}_r^t$. This is equivalent to the process of doing the above backwards induction assuming that $\bar{P}_r^\tau = \bar{P}_r^t, \forall \tau > t$. This process would result in the same set of matrices $\tilde{A}_t, \tilde{G}_t$, etc, with only the constant terms such as $E_{T+1}[w_T^{'} \tilde{P}_T w_T]$ missing. So, Algorithm 1 makes the same decision $v(t)$ as in the optimal solution of DP2.

Next, we claim that Algorithm 1 also gives the maximal expected welfare to DP3, where DP3 is defined to be the same as DP2 except that the cost for real-time energy is changed to $c_b\left([Q(t) - P_r(t) - P_d(t)]_+; t\right)$. This is true because by assumption, $Q(t) - P_r(t) - P_d(t) > 0$ always holds during the execution of Algorithm 1. Therefore, the decisions made by Algorithm 1 satisfy the optimality conditions for DP3.

Finally, our DP considered in this proposition has more constraints (i.e., $P_d \geq 0$ and (7)) than DP3. Since Algorithm 1 gives the maximal expected welfare to DP3 without violating the extra constraints, it also gives the maximal expected welfare to our DP. ■

APPENDIX B
PROOF OF PROPOSITION 2

Proof: Consider an idealized algorithm which knows the realization of $P_r(t), \forall t$ in advance, and chooses P_d and $q_1(t), \forall t$ as follows to maximize the welfare.

$$\begin{aligned} W_{ideal}(P_r) \quad & := \\ \max_{P_d \geq 0, q_1} \quad & \sum_{t=1}^{T} [-c_d(P_d(t); t) - \\ & c_b([q_1(t) - P_d(t) - P_r(t)]_+; t)] \\ s.t. \quad & q_1(t) \geq 0, \sum_{t=1}^{T} q_1(t) \geq \overline{Q}_1. \qquad (18) \end{aligned}$$

Equivalently, this can be written as

$$\begin{aligned} W_{ideal}(P_r) \quad & := \\ \max_{P_d, P_b, q_1} \quad & \sum_{t=1}^{T} [-c_d(P_d(t); t) - c_b(P_b(t); t)] \\ s.t. \quad & q_1(t) \geq 0, \forall t, \sum_{t=1}^{T} q_1(t) \geq \overline{Q}_1, \\ & P_d, P_b \geq 0, \\ & P_r(t) + P_d(t) + P_b(t) \geq q_1(t), \forall t. (19) \end{aligned}$$

The idealized algorithm performs better than the optimal dynamic-programming solution (because for each realization of $\{P_r(t)\}$, the idealized algorithm performs better). From (19), it is easy to see that $W_{ideal}(P_r)$ is concave in P_r. So,

$$J^* \leq E(W_{ideal}(P_r)) \leq W_{ideal}(\bar{P}_r)$$

where $\bar{P}_r = E(P_r)$. Since $\sum_t \bar{P}_r(t) < \overline{Q}_1$ by assumption, we have

$$\begin{aligned} W_{ideal}(\bar{P}_r) \quad & = \quad -2T \cdot \frac{1}{2} ([\frac{\overline{Q}_1 - \sum_t \bar{P}_r(t)}{2T}]^2 \\ & = \quad -\frac{[\overline{Q}_1 - \sum_t \bar{P}_r(t)]^2}{4T}. \end{aligned}$$

Therefore

$$J^* \leq -\frac{[\overline{Q}_1 - \sum_t \bar{P}_r(t)]^2}{4T}. \qquad (20)$$

Now we provide a lower bound of J. First, note that in Algorithm 1, the day-ahead energy is chosen as $P_d^*(t) = \frac{\overline{Q}_1 - \sum_\tau \bar{P}_r(\tau)}{2T}, \forall t$. In slot t, the real-time energy is chosen as $P_b^*(t) = [\frac{\overline{Q}_1 - \sum_{\tau=1}^{t-1} q_1^*(\tau) - P_r(t) - \sum_{\tau=t+1}^{T} \bar{P}_r(\tau)}{T-t+1}]_+$ and $q_1^*(t) = P_r(t) + P_d^*(t) + P_b^*(t)$.

Consider an algorithm (called Algorithm 2) which follows the similar equations (where $P_d^{**}(t)$, $P_b^{**}(t)$ and $q_1^{**}(t)$ denote its decisions):

$$P_d^{**}(t) = \frac{\overline{Q}_1 - \sum_\tau \bar{P}_r(\tau)}{2T}, \forall t \tag{21}$$

and for $t = 1, 2, \ldots, T$,

$$P_b^{**}(t) = \frac{\overline{Q}_1 - \sum_{\tau=1}^{t-1} q_1^{**}(\tau) - P_r(t) - \sum_{\tau=t+1}^{T} \bar{P}_r(\tau)}{T-t+1}, \tag{22}$$

$$q_1^{**}(t) = P_r(t) + P_d^{**}(t) + P_b^{**}(t). \tag{23}$$

Note that the only difference from Algorithm 1 is that here we don't confine $P_b^{**}(t)$ to be non-negative, and we define $c_b(P;t) = \frac{1}{2}P^2$ for any $P \in (-\infty, \infty)$ for Algorithm 2.

We can show that J_2, the expected welfare achieved by Algorithm 2, satisfies that

$$J_2 \leq J. \tag{24}$$

For brevity, the proof is omitted. (But the idea is that, if in slot t Algorithm 2 chooses a negative $P_b^{**}(t)$, it incurs more cost in slot t than Algorithm 1 which chooses $P_b^*(t) = 0$, and also leaves more remaining demand than Algorithm 1.)

Also, by (21)–(23), we can compute that

$$P_b^{**}(t) = \frac{\overline{Q}_1 - \sum_t \bar{P}_r(t)}{2T} + \sum_{\tau=1}^{t} \frac{\bar{P}_r(\tau) - P_r(\tau)}{T-\tau+1}, \forall t.$$

So,

$$\begin{aligned} J_2 &= -E\{T\frac{1}{2}[\frac{\overline{Q}_1 - \sum_t \bar{P}_r(t)}{2T}]^2 + \\ &\quad \sum_{t=1}^{T} \frac{1}{2}[\frac{\overline{Q}_1 - \sum_t \bar{P}_r(t)}{2T} + \sum_{\tau=1}^{t} \frac{\bar{P}_r(\tau) - P_r(\tau)}{T-\tau+1}]^2\} \\ &= -\{T\frac{1}{2}[\frac{\overline{Q}_1 - \sum_t \bar{P}_r(t)}{2T}]^2 + T\frac{1}{2}[\frac{\overline{Q}_1 - \sum_t \bar{P}_r(t)}{2T}]^2 \\ &\quad + \sum_{\tau=1}^{T} \frac{\sigma^2(\tau)}{T-\tau+1}\} \\ &= -\frac{[\overline{Q}_1 - \sum_t \bar{P}_r(t)]^2}{4T} - \sum_{\tau=1}^{T} \frac{\sigma^2(\tau)}{T-\tau+1} \\ &\geq J^* - \sum_{\tau=1}^{T} \frac{\sigma^2(\tau)}{T-\tau+1} \end{aligned}$$

using (20). Combining this with (24), we complete the proof. ∎

APPENDIX C
PROOF OF PROPOSITION 3

Proof: (i) With b fixed, consider two constants $a_2 > a_1$. Assume that when $a = a_1$, the optimal policy is $\phi^{(a_1,b)}$. When $a = a_2$, we construct a policy $\phi^{'}$ according to $\phi^{(a_1,b)}$, as follows. Note that $P_r(t; a_2, b) > P_r(t; a_1, b)$. Let policy $\phi^{'}$ follow the same action of $\phi^{(a_1,b)}$, pretending that the renewable energy is $P_r(t; a_1, b)$ (which is smaller than the actual $P_r(t; a_2, b)$). Then the expected welfare achieved by policy $\phi^{'}$ is at least $J^*(a_1, b)$. So, the optimal policy $\phi^{(a_2,b)}$ when $a = a_2$ achieves an expected welfare not less than $J^*(a_1, b)$. This completes the proof.

(ii) With a fixed, consider $b_2 > b_1$. For simplicity, denote the value function with renewable energy $P_r(t; a, b_1)$ by $J_t(x(t))$, and the value function with $P_r(t; a, b_2)$ by $\bar{J}_t(x(t))$. The Bellman equations for $J_t(x(t))$ are (where $t = 0$ means "day-ahead")

$$\begin{aligned} J_0(x(0)) &= \max_{v(0)\geq 0} \{W_0^v(x(0), v(0)) + E[J_1(x(0) \\ &+ M_{v,0} \cdot v(0) + M_r P_r(1; a, b_1) + h_0)]\} \end{aligned} \tag{25}$$

$$\begin{aligned} &J_t(x(t)) = \\ &\max_{v(t)\in[\underline{q}(t), \overline{q}(t)]} \{W_t^v(x(t), v(t)) + E[J_{t+1}(M_x \cdot x(t) \\ &-M_v \cdot v(t) + M_r P_r(t+1; a, b_1))]\}, \forall 1 \leq t < T \end{aligned} \tag{26}$$

where $W_t^v(x(t), v(t)), t = 0, 1, \ldots, T$ is defined in (11) and (12), the constant matrices $M_{v,0}, M_r M_x, M_v$ and vector h_0 correspond to those in the RHS of (9) and (10). We define that

$$P_r(T+1) := 0; \; J_{T+1}(.) = -c_{T+1}(\cdot).$$

Similar equations hold for $\bar{J}_t(x(t))$.

We will prove, by induction, that for any $0 \leq t \leq T$

$$J_t(x(t)) \geq \bar{J}_t(x(t)), \tag{27}$$

and that $J_t(\cdot)$ and $\bar{J}_t(\cdot)$ are concave. First, it is not difficult to see that $J_T(x(T)) = \bar{J}_T(x(T))$ (therefore (27) holds for $t = T$), and $J_T(\cdot), \bar{J}_T(\cdot)$ are concave. Assume that (27) holds for $2 \leq t+1 \leq T$, and $J_{t+1}(\cdot), \bar{J}_{t+1}(\cdot)$ are concave, we show that these also hold for t. To see this, note that

$$\begin{aligned} & J_{t+1}(M_x x(t) - M_v v(t) + M_r P_r(t+1; a, b_1)) \\ \geq \; & \bar{J}_{t+1}(M_x x(t) - M_v v(t) + M_r P_r(t+1; a, b_1)). \end{aligned}$$

So

$$\begin{aligned} & E\{J_{t+1}(M_x x(t) - M_v v(t) + M_r P_r(t+1; a, b_1))\} \\ \geq \; & E\{\bar{J}_{t+1}(M_x x(t) - M_v v(t) + M_r P_r(t+1; a, b_1))\} \\ \geq \; & E[\bar{J}_{t+1}(M_x x(t) - M_v v(t) + M_r P_r(t+1; a, b_2))] \end{aligned}$$

where the first "$\geq$" follows from the previous inequality, and the second "$\geq$" holds because $\bar{J}_{t+1}(\cdot)$ is concave, and $b_2 > b_1$. Using this relation and Eq. (26) (and its analogy for $\bar{J}_t(r_t)$), we know that (27) holds for $t \geq 1$.

Note that in (26), the expression inside "max" is concave in $(x(t), v(t))$. Therefore $J_t(\cdot)$ is concave, and so is $\bar{J}_t(\cdot)$.

The above results similarly hold for $t = 0$. Therefore, $J^*(a_1, b) = J_0(0_{T+N+1}) \geq \bar{J}_0(0_{T+N+1}) = J^*(a_2, b)$.

(iii) Since $\mu_r(t) + V_r(t) \geq 0$, we know that if $s_2 > s_1 \geq 0$, then $P_r(t; s_2, s_2) \geq P_r(t; s_1, s_1)$. The rest of the proof is similar to (i). ∎

REFERENCES

[1] L. Chen, N. Li, L. Jiang, and S. H. Low. Optimal demand response: problem formulation and deterministic case. *submitted as a book chapter in Control and Optimization Theory for Electric Smart Grids*, 2011, Available: http://www.its.caltech.edu/~libinj/Springer.pdf.

[2] L. Jiang and S. H. Low. Multi-period optimal energy procurement and demand response in smart grid with uncertain supply. In *IEEE Conference on Decision and Control*, December 2011.