Optimal Decentralized Primary Frequency Control in Power Networks

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Abstract— **We augment existing generator-side primary frequency control with load-side control that are local, ubiquitous, and continuous. The mechanisms on both the generator and the load sides are decentralized in that their control decisions are functions of locally measurable frequency deviations. These local algorithms interact over the network through nonlinear power flows. We design the local frequency feedback control so that any equilibrium point of the closed-loop system is the solution to an optimization problem that minimizes the total generation cost and user disutility subject to power balance across entire network. With Lyapunov method we derive a sufficient condition for any equilibrium point of the closed-loop system to be asymptotically stable. A simulation demonstrates improvement in both the transient and steady-state performance over the traditional control only on generators, even when the total control capacity remains the same.**

I. INTRODUCTION

It is important to maintain the frequency of a power system tightly around its nominal value for the quality of power delivery and safety of infrastructures. Frequency is mainly determined by real power imbalance through dynamics of rotating units in the system, and hence frequency is usually controlled by adjusting real power injections. Frequency/real power control is traditionally implemented on the generator side and usually consists of three different layers that work at different timescales in concert as described in [1]–[3]. As generation or load fluctuates, the primary frequency control operates continuously to stop frequency excursion. Generator-side primary frequency control is also known as droop control, in which a speed governor adjusts the generation power based on local frequency feedback. The secondary frequency control, also known as automatic generation control (AGC), operates at time steps of several seconds and adjusts the setpoints of governors in a control area in a centralized fashion to drive the frequency back to its nominal value and the inter-area power flows to their scheduled values. Economic dispatch, also known as the tertiary control, operates at time steps of several minutes or up and schedules the output levels of generators that are online and the power flows.

As a supplement to generator-side frequency control, ubiquitous continuous distributed load participation in frequency control has started to play a rising role since the last decade or so. Household appliances such as refrigerators, heaters, ventilation systems, air conditioners, and plug-in electric vehicles can be controlled to help manage power imbalance and regulate frequency. This idea dates back to the late 1970s [4], and has been extensively studied recently [5]–[7], with a particular focus on primary frequency control [8]–[10]. There also have been field trials around the world [11]–[13]. Simulation-based studies and field trials above have shown significant improvement in performance mainly due to fast-acting capability of frequency-responsive loads and reduction in the need for generator reserves with high emissions. In this paper, we provide a systematic method to jointly design generator and load-side primary frequency control.

Early work on the design of generator-side primary frequency control focuses on stabilizing multi-machine power networks [14]–[19]. There were also numerous papers, e.g., [20]–[22], on stability analysis of power networks with linear frequency dependent loads, which laid the foundation for studying the stability of load-side primary frequency control. As renewable generation introduces larger and faster fluctuations in real power and frequency, recent studies integrate functions traditionally realized by slower-timescale control, e.g, economic dispatch, with faster-timescale control, e.g., primary frequency control. Examples of these studies range from primary and/or secondary frequency control on the generator side [23]–[28], or the load side [29]–[32], to microgrids where inverters have similar dynamic behavior to generators [33], [34]. The control schemes in all these recent studies are decentralized or distributed, and hence are scalable to networks with a large number of controllable endpoints and suitable for deployment in future power grids.

Compared to the work above, our work in this paper (i) designs primary frequency control jointly on the generator and load sides, (ii) considers a more realistic generator model including governor and turbine dynamics, (iii) designs a completely decentralized control scheme in which every generator and load makes its decision based on local frequency sensing and its own cost of generation or disutility for participating in load control, (iv) attains equilibrium points characterized by an optimization problem called *optimal frequency control* (OFC), which minimizes the total generation cost plus total user disutility while maintaining the power balance across entire network, (v) applies Lyapunov method to the dynamical network model with nonlinear real power flows to derive a sufficient condition for asymptotic stability of equilibrium points, and (vi) does not rely on any specific network topology like tree [25], [33], [34] or star [24]. We note that different subsets of (i)–(vi) are also achieved in some of the papers [14]–[34]. A simulation with a more realistic model shows that the proposed control improves

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both the transient and steady-state frequency, compared with traditional control on generators only, with the same total control capacity.

The rest of this paper is organized as follows. Section II describes a dynamical model of power networks. Section III formulates the OFC problem which characterizes the desired equilibria and guides the design of decentralized primary frequency control. Section IV derives a sufficient condition for any equilibrium point of the closed-loop system to be asymptotically stable. Section V is a simulation-based case study to show the performance of the proposed control. Section VI concludes the paper and discusses future work.

II. POWER NETWORK MODEL

Let $\mathbb{R}$ denote the set of real numbers. For a set $\mathcal{N}$, let $|\mathcal{N}|$ denote its cardinality. A variable without a subscript usually denotes a vector with appropriate components, e.g., $\omega = (\omega_j, j \in \mathcal{N}) \in \mathbb{R}^{|\mathcal{N}|}$. For $a, b \in \mathbb{R}$, $a \le b$, the expression $[\cdot]_a^b$ denotes $\max\{\min\{\cdot,\ b\},\ a\}$. For a matrix A, let A^T denote its transpose. For a square matrix A, the expression $A \succ 0$ ($A \prec 0$) means it is positive (negative) definite. For a signal $\omega(t)$ of time t, let $\dot{\omega}$ denote its time derivative $d\omega/dt$.

We combine the classical structure preserving model in [20] and the generator speed governor and turbine models in [1]–[3], [16]–[18]. The power network is modeled as a graph $(\mathcal{N}, \mathcal{E})$ where $\mathcal{N} = \{1, \ldots, |\mathcal{N}|\}$ is the set of buses and $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is the set of lines connecting those buses. We use (i, j) to denote the line connecting buses $i, j \in \mathcal{N}$, and assume that $(\mathcal{N}, \mathcal{E})$ is directed, with an arbitrary orientation, so that if $(i, j) \in \mathcal{E}$ then $(j, i) \notin \mathcal{E}$. We use "$i : i \to j$" and "$k : j \to k$" respectively to denote the set of buses i that are predecessors of bus j and the set of buses k that are successors of bus j. We assume without loss of generality that $(\mathcal{N}, \mathcal{E})$ is connected, and make the following assumptions which are well-justified for transmission networks:

- The lines $(i, j) \in \mathcal{E}$ are lossless and characterized by their reactances x_{ij}.
- The voltage magnitudes $|V_j|$ of buses $j \in \mathcal{N}$ are constants.
- Reactive power injections on buses and reactive power flows on lines are not considered.

A subset $\mathcal{G} \in \mathcal{N}$ of the buses are fictitious buses representing the internal of generators. Hence we call $\mathcal{G}$ the set of generators and $\mathcal{L} := \mathcal{N} \backslash \mathcal{G}$ the set of load buses.[1] We label the buses so that $\mathcal{G} = \{1, \ldots, |\mathcal{G}|\}$ and $\mathcal{L} = \{|\mathcal{G}| + 1, \ldots, |\mathcal{N}|\}$.

The voltage phase angle of bus $j \in \mathcal{N}$, with respect to the rotating framework of nominal frequency $\omega_0 = 120\pi$ rad/s, is denoted by θ_j. Let ω_j be the frequency deviation of bus j from the nominal frequency ω_0. Hence we have

$$\dot{\theta}_j = \omega_j \qquad j \in \mathcal{N}. \tag{1}$$

The system dynamics are described by the swing equations

$$M_j \dot{\omega}_j = -D_j \omega_j + p_j - F_j(\theta) \qquad j \in \mathcal{N} \tag{2}$$

[1] We call all the buses in $\mathcal{L}$ load buses without distinguishing between generator buses (buses connected directly to generators) and actual load buses, since they are treated in the same way mathematically.

where $M_j > 0$ for $j \in \mathcal{G}$ are moments of inertia of generators and $M_j = 0$ for $j \in \mathcal{L}$, and $D_j > 0$ for all $j \in \mathcal{N}$ are (for $j \in \mathcal{G}$) the damping coefficients of generators or (for $j \in \mathcal{L}$) the coefficients of linear frequency dependent loads, e.g., induction motors. The variable p_j denotes the real power injection to bus j, which is the mechanic power injection to generator if $j \in \mathcal{G}$, and is the negative of real power load if $j \in \mathcal{L}$. The net real power flow out of bus j is

$$F_j(\theta) := \sum_{k: j \to k} Y_{jk} \sin(\theta_j - \theta_k) - \sum_{i: i \to j} Y_{ij} \sin(\theta_i - \theta_j) \qquad j \in \mathcal{N} \tag{3}$$

where $Y_{jk} := \frac{|V_j||V_k|}{x_{jk}}$ are the maximum real power flows on lines $(j, k) \in \mathcal{E}$.

Associated with a generator $j \in \mathcal{G}$ is a system of governor and turbine. From [1], their dynamics are described by

$$\dot{a}_j = -\frac{1}{\tau_{g,j}} a_j + \frac{1}{\tau_{g,j}} p_j^c \qquad j \in \mathcal{G} \tag{4}$$

$$\dot{p}_j = -\frac{1}{\tau_{b,j}} p_j + \frac{1}{\tau_{b,j}} a_j \qquad j \in \mathcal{G} \tag{5}$$

where a_j is the valve position of the turbine, p_j^c is the control command to the generator, and p_j, as introduced above, is the mechanic power injection to the generator. The time constants $\tau_{g,j}$ and $\tau_{b,j}$ characterize respectively the time-delay in governor action and the approximated fluid dynamics in the turbine. Traditionally, there is a frequency feedback term $-\frac{1}{R_j}\omega_j$ added to the right-hand-side of (4), known as the frequency droop control. Here this term is merged into p_j^c to allow for a general form of frequency feedback control.

Equations (1)–(5) specify a dynamical system with state variables $(\theta, \omega, a_{\mathcal{G}}, p_{\mathcal{G}})$ where

$$\theta := \{\theta_1, \ldots, \theta_{|\mathcal{N}|}\}, \quad \omega := \{\omega_1, \ldots, \omega_{|\mathcal{N}|}\}$$
$$a_{\mathcal{G}} := \{a_1, \ldots, a_{|\mathcal{G}|}\}, \quad p_{\mathcal{G}} := \{p_1, \ldots, p_{|\mathcal{G}|}\}$$

and input variables $(p_{\mathcal{G}}^c, p_{\mathcal{L}})$ where

$$p_{\mathcal{G}}^c := \{p_1^c, \ldots, p_{|\mathcal{G}|}^c\}, \quad p_{\mathcal{L}} := \{p_{|\mathcal{G}+1|}, \ldots, p_{|\mathcal{N}|}\}.$$

In the sequel, $(p_{\mathcal{G}}^c, p_{\mathcal{L}})$ are feedback control to be designed based on local measurements of frequency deviations, i.e., $(p_{\mathcal{G}}^c(\omega), p_{\mathcal{L}}(\omega))$. Parameters $\underline{p}_j \le \overline{p}_j$ specify the bounds of the control variables, i.e., $\underline{p}_j \le p_j^c(\omega) \le \overline{p}_j$ for $j \in \mathcal{G}$, and $\underline{p}_j \le p_j(\omega) \le \overline{p}_j$ for $j \in \mathcal{L}$. Note that if $\underline{p}_j = \overline{p}_j$ then they specify a constant, uncontrollable input on bus j. To motivate the work below, we first define equilibrium points for this closed-loop dynamical system.

Definition 1: An equilibrium point of the system (1)–(5) with control $(p_{\mathcal{G}}^c(\omega), p_{\mathcal{L}}(\omega))$, or a *closed-loop equilibrium* for short, is $(\theta^*, \omega^*, a_{\mathcal{G}}^*, p_{\mathcal{G}}^*, p_{\mathcal{G}}^{c,*}, p_{\mathcal{L}}^*)$, where θ^* is a vector function of time and $(\omega^*, a_{\mathcal{G}}^*, p_{\mathcal{G}}^*, p_{\mathcal{G}}^{c,*}, p_{\mathcal{L}}^*)$ are vectors of real numbers, such that

$$p_{\mathcal{G}}^{c,*} = p_{\mathcal{G}}^c(\omega^*), \quad p_{\mathcal{L}}^* = p_{\mathcal{L}}(\omega^*) \tag{6}$$

$$d\theta_j^*/dt = \omega_j^* \qquad j \in \mathcal{N} \tag{7}$$

$$\omega_i^* = \omega_j^* = \omega^* \qquad i,j \in \mathcal{N}\ ^2 \tag{8}$$

$$-D_j\omega_j^* + p_j^* - F_j(\theta^*) = 0 \qquad j \in \mathcal{N} \tag{9}$$

$$p_j^* = a_j^* = p_j^{c,*} \qquad j \in \mathcal{G}. \tag{10}$$

In the definition above, (8) ensures constant $F(\theta^*)$ at equilibrium points by (3), and (9)(10) are obtained by letting right-hand-sides of (2)(4)(5) be zero. From (8), at any equilibrium point, all the buses are synchronized to the same frequency. The system typically has multiple equilibrium points as will be explained in Section IV. We also write an equilibrium point as $(\theta^*, \omega^*, a_{\mathcal{G}}^*, p_{\mathcal{G}}^{c,*}, p^*)$ where $p^* := (p_{\mathcal{G}}^*, p_{\mathcal{L}}^*)$, when we do not have to distinguish between state variables $p_{\mathcal{G}}^*$ and control variables $p_{\mathcal{L}}^*$.

III. DECENTRALIZED PRIMARY FREQUENCY CONTROL

An initial point of the dynamical system (1)–(5) corresponds to the state of the system at the time of fault-clearance after a contingency, or the time at which an unscheduled change in power injection occurs during normal operation. In either case, it is required that the system trajectory, driven by primary frequency control $(p_{\mathcal{G}}^c(\omega), p_{\mathcal{L}}(\omega))$, converges to a *desired* equilibrium point. In this section, we formalize the criteria for desired equilibrium points by formulating an optimization problem called *optimal frequency control* (OFC), and use OFC to guide the design of control. In Section IV we will study the stability of the closed-loop system with the proposed control.

A. Optimal Frequency Control Problem

Our objective is to rebalance power after a disturbance at a minimum generation cost and user disutility. This is formalized by requiring any closed-loop equilibrium (p^*, d^*) to be a solution of the following OFC problem, where $d_j^* = D_j\omega_j^*$ for $j \in \mathcal{N}$.

OFC:

$$\min_{p,d} \quad \sum_{j\in\mathcal{N}} \left(c_j(p_j) + \frac{1}{2D_j} d_j^2 \right) \tag{11}$$

$$\text{subject to} \quad \sum_{j\in\mathcal{N}} (p_j - d_j) = 0 \tag{12}$$

$$\underline{p}_j \le p_j \le \overline{p}_j \qquad j \in \mathcal{N}. \tag{13}$$

The term $c_j(p_j)$ in objective function (11) is generation cost (if $j \in \mathcal{G}$) or user disutility for participating in load control (if $j \in \mathcal{L}$). For simplicity we call c_j a cost function for $j \in \mathcal{N}$ even if it may be a user disutility function. The term $\frac{1}{2D_j} d_j^2$ implicitly penalizes frequency deviation on bus j at equilibrium. More detail for the motivation of (11), such as why the weighting factor of the second term is selected as $\frac{1}{2D_j}$, can be found in [30]. The constraint (12) ensures power balance over entire network, and (13) are bounds on power injections. These bounds are determined by control capacities of generators or controllable loads, as well as uncontrollable power injections as an exogenous input.

[2] We abuse the notation by using ω^* to denote both the vector $(\omega_1^*, \ldots, \omega_{|\mathcal{N}|}^*)$ and the common value of its components. Its meaning should be clear from the context.

We assume Conditions 1 and 2 below throughout this paper.

Condition 1: OFC is feasible. The cost functions c_j are strictly convex and twice continuously differentiable on $(\underline{p}_j, \overline{p}_j)$.

Remark 1: A load $-p_j$ on bus $j \in \mathcal{L}$ results in a user utility $u_j(-p_j)$, and hence the disutility function can be defined as $c_j(p_j) = -u_j(-p_j)$. In the literature on demand response [35] and economic dispatch [3], [23]–[25], [27], [34], the user disutility functions or generation cost functions usually satisfy Condition 1, and in many cases are quadratic functions. See [30, Sec. III-A] for more references that use cost functions that satisfy Condition 1.

Condition 2: For any optimal solution (p^*, d^*) of OFC, the power flow equations

$$F_j(\theta) = p_j^* - d_j^* \qquad j \in \mathcal{N} \tag{14}$$

are feasible, i.e., have at least one solution $\theta^* \in \mathbb{R}^{|\mathcal{N}|}$.

Remark 2: A lot of work, e.g., [36], studies the feasibility of power flow equations (14), which is beyond the scope of this paper. As will become clear below, Condition 2 ensures the existence of a closed-loop equilibrium of the dynamical system (1)–(5) with the feedback control proposed below.

B. Design of Decentralized Feedback Control

We use OFC to guide our controller design. We now specify our design and then prove that any resulting closed-loop equilibrium is the unique solution of the OFC problem. Similar to [30], we design $(p_{\mathcal{G}}^c(\omega), p_{\mathcal{L}}(\omega))$ as

$$p_j^c(\omega_j) = \left[(c_j')^{-1}(-\omega_j)\right]_{\underline{p}_j}^{\overline{p}_j} \qquad j \in \mathcal{G} \tag{15}$$

$$p_j(\omega_j) = \left[(c_j')^{-1}(-\omega_j)\right]_{\underline{p}_j}^{\overline{p}_j} \qquad j \in \mathcal{L}. \tag{16}$$

The function $(c_j')^{-1}(\cdot)$, which is the inverse function of the derivative of the cost function, is well defined if Condition 1 holds. Note that this control is completely decentralized in that for every generator and load indexed by j, the control decision is a function of frequency deviation ω_j measured at its local bus. Only its own cost function c_j and bounds $[\underline{p}_j, \overline{p}_j]$ need to be known. No explicit communication with other generators and loads is required, nor is any knowledge of system parameters. Moreover, the following theorem shows that this design fulfills our objective.

Theorem 1: Suppose Conditions 1 and 2 hold. Then we have both of the following:

1) There is a *unique* optimal solution of OFC and its dual.
2) For the system (1)–(5) with control (15)(16), there exists at least one equilibrium point. Let $(\theta^*, \omega^*, a_{\mathcal{G}}^*, p_{\mathcal{G}}^{c,*}, p^*)$ be any of its equilibrium point(s). Then $(p^*, d^*; \omega^*)$ is the *unique* optimal solution of OFC and its dual, where $d_j^* = D_j\omega^*$ for $j \in \mathcal{N}$.

Proof: Denote the variables of the dual of OFC by (λ, μ^+, μ^-), where $\lambda \in \mathbb{R}$ and μ^+ and μ^- are both vectors in $\mathbb{R}^{|\mathcal{N}|}$. Since OFC is a feasible (by Condition 1) convex problem with differentiable objective function and affine inequality constraints, by [37], Slater's theorem states that strong duality holds and there exists a feasible dual optimal solution. Moreover, a point $(p, d; \lambda, \mu^+, \mu^-)$ is optimal for OFC and its dual if and only if it satisfies the following Karus-Kuhn-Tucker (KKT) condition:

$$c_j'(p_j) + \lambda + \mu_j^+ - \mu_j^- = 0 \qquad j \in \mathcal{N} \tag{17}$$
$$d_j - \lambda D_j = 0 \qquad j \in \mathcal{N} \tag{18}$$
$$\sum_{j \in \mathcal{N}} (p_j - d_j) = 0 \tag{19}$$
$$\underline{p}_j \le p_j \le \overline{p}_j \qquad j \in \mathcal{N} \tag{20}$$
$$\mu_j^+ \ge 0, \quad \mu_j^- \ge 0 \qquad j \in \mathcal{N} \tag{21}$$
$$\mu_j^+(p_j - \overline{p}_j) = \mu_j^-(\underline{p}_j - p_j) = 0 \quad j \in \mathcal{N} \tag{22}$$

where (17)(18) are stationarity conditions, (19)(20) are primal feasibility, (21) is dual feasibility and (22) is complementary slackness. With (μ^+, μ^-) eliminated, (17)–(22) are equivalent to (18)(19) and the following equation

$$p_j = \left[(c_j')^{-1}(-\lambda)\right]_{\underline{p}_j}^{\overline{p}_j} \qquad j \in \mathcal{N}. \tag{23}$$

Note that (18)(19)(23) have a unique solution $(p^0, d^0; \lambda^0)$ since a primal-dual optimal solution exists and, by Condition 1, the left-hand-side of (19) is a strictly decreasing function of λ. On the other hand, any closed-loop equilibrium $(\theta^*, \omega^*, a_{\mathcal{G}}^*, p_{\mathcal{G}}^{c,*}, p^*)$, if exists, satisfies

$$d_j^* = D_j \omega^* \qquad j \in \mathcal{N} \tag{24}$$
$$\sum_{j \in \mathcal{N}} (p_j^* - d_j^*) = 0 \tag{25}$$
$$p_j^* = \left[(c_j')^{-1}(-\omega^*)\right]_{\underline{p}_j}^{\overline{p}_j} \qquad j \in \mathcal{N} \tag{26}$$

where (25) results from adding up (9) for all $j \in \mathcal{N}$ to eliminate $F_j(\theta)$, and (26) is a result of (10) and the control (15)(16). Hence $(p^*, d^*; \omega^*) = (p^0, d^0; \lambda^0)$ is the unique solution of (18)(19)(23) and therefore the unique optimal solution of OFC and its dual (ignoring μ^+, μ^- which may not be unique at optimal). Moreover, by Condition 2, there exists at least one solution $\theta^* \in \mathbb{R}^{|\mathcal{N}|}$ for (14). Therefore $(\omega^* t + \theta^*, \omega^*, a_{\mathcal{G}}^*, p_{\mathcal{G}}^{c,*}, p^*)$, where $a_{\mathcal{G}}^* = p_{\mathcal{G}}^{c,*} = p_{\mathcal{G}}^*$, is a closed-loop equilibrium, by Definition 1. ∎

Remark 3: In general ω^* is not zero, which means the frequency is not restored to the nominal value at equilibrium. Recovery to nominal frequency needs secondary frequency control [1]–[3], which is traditionally centralized within a control area; see [26], [31], [32], [34] for recent work on distributed secondary frequency control.

IV. STABILITY OF CLOSED-LOOP EQUILIBRIA

We use Lyapunov method to study the stability of the closed-loop system, and derive a sufficient condition for any closed-loop equilibrium to be asymptotically stable. Our approach for stability analysis is compositional [38], in that we find Lyapunov function candidates separately for the network with load control and the generators, and add them up to form a Lyapunov function.

Theorem 1 implies the uniqueness of $(\omega^*, a_{\mathcal{G}}^*, p_{\mathcal{G}}^{c,*}, p^*)$ across all the equilibrium points of the system (1)–(5) with control (15)(16). However, θ^* may not be unique even if we ignore any difference by multiples of 2π and regard two solutions θ^* and $\hat{\theta}$ to (14) as the same if $\theta_j^* - \hat{\theta}_j$ are the same for all $j \in \mathcal{N}$ [36]. Due to the possible existence of multiple closed-loop equilibria, we focus on their *local* asymptotic stability. In particular, given any closed-loop equilibrium $(\theta^*, \omega^*, a_{\mathcal{G}}^*, p_{\mathcal{G}}^{c,*}, p^*)$, we derive a sufficient condition for asymptotic stability, as Conditions 3 and 4 below.

Condition 3: $\left|\theta_i^* - \theta_j^*\right| < \frac{\pi}{2}$ for all $(i, j) \in \mathcal{E}$.

Remark 4: Though θ_j^* for $j \in \mathcal{N}$ are functions of time by Definition 1, their differences $\theta_i^* - \theta_j^*$ for all $i, j \in \mathcal{N}$ are constant, by (7)(8). In practice Condition 3 is often considered as a security constraint for power flow solutions [34]. Indeed it guarantees the asymptotic stability of the equilibrium (θ^*, ω^*) of the open-loop system (1)–(3) with constant power input p^* [20].

Condition 4: For all $j \in \mathcal{G}$, there exists a constant L_j which satisfies $0 \le L_j < D_j$, such that the generator controller design $p_j^c(\cdot)$ in (15) satisfies

$$|p_j^c(\omega_j) - p_j^c(\omega^*)| \le L_j |\omega_j - \omega^*|$$

for all ω_j in a neighborhood of ω^*.

Theorem 2: Suppose Conditions 1 and 2 hold so that so closed-loop equilibria exist. Then any closed-loop equilibrium $(\theta^*, \omega^*, a_{\mathcal{G}}^*, p_{\mathcal{G}}^{c,*}, p^*)$ of the system (1)–(5) with control (15)(16) is locally asymptotically stable, if it satisfies Conditions 3 and 4.

Proof: We use the following energy function, which was used in [20] for stability analysis of the open-loop system (1)–(3), as part of the Lyapunov function candidate for our closed-loop system. Given a closed-loop equilibrium $(\theta^*, \omega^*, a_{\mathcal{G}}^*, p_{\mathcal{G}}^{c,*}, p^*)$ which satisfies Conditions 3 and 4, the energy function is

$$\begin{aligned}\mathcal{V}_0 = &\frac{1}{2}\sum_{j \in \mathcal{G}} M_j(\omega_j - \omega_j^*)^2 \\ &+ \sum_{(i,j) \in \mathcal{E}} \int_{\theta_{ij}^*}^{\theta_{ij}} Y_{ij}(\sin u - \sin \theta_{ij}^*) du\end{aligned} \tag{27}$$

where $\theta_{ij} := \theta_i - \theta_j$. By Condition 3, the integral term for every $(i, j) \in \mathcal{E}$ in (27) is positive definite in a neighborhood of θ_{ij}^*, and is zero only when $\theta_{ij} = \theta_{ij}^*$. For simplicity define $\tilde{\omega} := \omega - \omega^*$ and use similar notations for deviations of other variables from their equilibrium values. Taking time derivative of (27) along any trajectory of (θ, ω), we have

$$\begin{aligned}\dot{\mathcal{V}}_0 &= \sum_{j\in\mathcal{G}} M_j \tilde{\omega}_j \dot{\omega}_j + \sum_{(i,j)\in\mathcal{E}} Y_{ij}(\sin\theta_{ij} - \sin\theta_{ij}^*)(\omega_i - \omega_j) \\ &= \sum_{j\in\mathcal{N}} \tilde{\omega}_j (p_j - D_j\omega_j - F_j(\theta^*)) \quad (28) \\ &= -\sum_{j\in\mathcal{N}} D_j\tilde{\omega}_j^2 + \sum_{j\in\mathcal{N}} \tilde{\omega}_j \tilde{p}_j \quad (29) \\ &\le -\sum_{j\in\mathcal{L}} D_j\tilde{\omega}_j^2 + \sum_{j\in\mathcal{G}} (-D_j\tilde{\omega}_j^2 + \tilde{\omega}_j\tilde{p}_j) \quad (30)\end{aligned}$$

where the equality in (28) results from (2)(3), the equality in (29) results from replacing $F_j(\theta^*)$ with $p_j^* - D_j\omega_j^*$ (by (9)), and the inequality in (30) holds since

$$\tilde{\omega}_j\tilde{p}_j = (\omega_j - \omega_j^*)(p_j(\omega_j) - p_j(\omega_j^*)) \le 0 \qquad j \in \mathcal{L}$$

where $p_j(\omega_j)$ is a non-increasing function of ω_j by (16).

We now study the other parts of the Lyapunov function candidate for generators $j \in \mathcal{G}$. By moving the origin of (4)(5) to the given closed-loop equilibrium, we have

$$\dot{\tilde{y}}_j = A_j\tilde{y}_j + B_j\tilde{p}_j^c \qquad j \in \mathcal{G} \quad (31)$$

where $\tilde{y}_j := [\tilde{a}_j, \tilde{p}_j]^T = [a_j - a_j^*, p_j - p_j^*]^T$, and $\tilde{p}_j^c := p_j^c(\omega_j) - p_j^c(\omega^*)$, and

$$A_j := \begin{bmatrix} -\frac{1}{\tau_{g,j}} & 0 \\ \frac{1}{\tau_{b,j}} & -\frac{1}{\tau_{b,j}} \end{bmatrix} \qquad B_j := \begin{bmatrix} \frac{1}{\tau_{g,j}} \\ 0 \end{bmatrix}.$$

A classical Lyapunov function candidate for the linear system (31) takes the form

$$\mathcal{V}_j = \frac{1}{2}\tilde{y}_j^T P_j \tilde{y}_j \qquad j \in \mathcal{G}$$

where P_j is a positive definite matrix. Hence the time derivative of $\mathcal{V}_j$ along any system trajectory is

$$\dot{\mathcal{V}}_j = \frac{1}{2}\tilde{y}_j^T(P_jA_j + A_j^TP_j)\tilde{y}_j + \tilde{y}_j^TP_jB_j\tilde{p}_j^c \qquad j \in \mathcal{G}. \quad (32)$$

Since both eigenvalues of A_j are negative, Lyapunov theory tells us that we can find $P_j \succ 0$ such that $P_jA_j + A_j^TP_j \prec 0$. Indeed, if for all $j \in \mathcal{G}$ we can find $P_j \succ 0$ such that

$$\dot{\mathcal{V}}_j \le -\alpha_j\tilde{p}_j^2 + \beta_j\tilde{\omega}_j^2 - \gamma_j(\tilde{a}_j + \eta_j\tilde{p}_j)^2 \qquad j \in \mathcal{G} \quad (33)$$

where $\alpha_j, \gamma_j > 0$, $\beta_j < D_j$, η_j is an arbitrary real number, and $4\alpha_j(D_j - \beta_j) > 1$, then, by (28)–(30), the time derivative of $\mathcal{V}_{\text{total}} := \mathcal{V}_0 + \sum_{j\in\mathcal{G}}\mathcal{V}_j$ satisfies

$$\begin{aligned}\dot{\mathcal{V}}_{\text{total}} \le &-\sum_{j\in\mathcal{L}} D_j\tilde{\omega}_j^2 - \sum_{j\in\mathcal{G}}\gamma_j(\tilde{a}_j + \eta_j\tilde{p}_j)^2 \\ &+ \sum_{j\in\mathcal{G}}(-(D_j - \beta_j)\tilde{\omega}_j^2 + \tilde{\omega}_j\tilde{p}_j - \alpha_j\tilde{p}_j^2)\end{aligned}$$

where the third summation is non-positive and is zero only when $\tilde{\omega}_j = \tilde{p}_j = 0$ for all $j \in \mathcal{G}$. It is straightforward that $\mathcal{V}_{\text{total}} \ge 0$ and $\dot{\mathcal{V}}_{\text{total}} \le 0$ in a neighborhood of the given closed-loop equilibrium, and both of them are zero only at the closed-loop equilibrium. Hence, to prove the given closed-loop equilibrium is asymptotically stable, it is sufficient to find $P_j \succ 0$ for all $j \in \mathcal{G}$ such that (33) holds.

We choose P_j to be diagonal with positive entries $P_{j,11}$ and $P_{j,22}$. To ensure $P_jA_j + A_j^TP_j \prec 0$, we have

$$\frac{P_{j,11}}{\tau_{g,j}} > \frac{P_{j,22}}{4\tau_{b,j}}.$$

A calculation from (32) gives

$$\begin{aligned}\dot{\mathcal{V}}_j &= -\frac{P_{j,11}}{\tau_{g,j}}\tilde{a}_j^2 - \frac{P_{j,22}}{\tau_{b,j}}\tilde{p}_j^2 + \frac{P_{j,22}}{\tau_{b,j}}\tilde{a}_j\tilde{p}_j + \frac{P_{j,11}}{\tau_{g,j}}\tilde{a}_j\tilde{p}_j^c \\ &= -(\frac{P_{j,22}}{\tau_{b,j}} - \frac{P_{j,22}^2}{4\gamma_j\tau_{b,j}^2})\tilde{p}_j^2 + \frac{P_{j,11}^2}{4\tau_{g,j}(P_{j,11} - \gamma_j\tau_{g,j})}(\tilde{p}_j^c)^2 \\ &\quad -\gamma_j\left(\tilde{a}_j - \frac{P_{j,22}}{2\gamma_j\tau_{b,j}}\tilde{p}_j\right)^2 \\ &\quad -(\frac{P_{j,11}}{\tau_{g,j}} - \gamma_j)\left(\tilde{a}_j - \frac{P_{j,11}\cdot\tilde{p}_j^c}{2(P_{j,11} - \gamma_j\tau_{g,j})}\right)^2 \quad (34)\end{aligned}$$

for arbitrary $\gamma_j \in (\frac{P_{j,22}}{4\tau_{b,j}}, \frac{P_{j,11}}{\tau_{g,j}})$. By Condition 4 we have $(\tilde{p}_j^c)^2 \le L_j^2\tilde{\omega}_j^2$ in the neighborhood referred to. Take

$$\alpha_j = \frac{P_{j,22}}{\tau_{b,j}} - \frac{P_{j,22}^2}{4\gamma_j\tau_{b,j}^2} \qquad \beta_j = \frac{P_{j,11}^2L_j^2}{4\tau_{g,j}(P_{j,11} - \gamma_j\tau_{g,j})}$$

$$\eta_j = -\frac{P_{j,22}}{2\gamma_j\tau_{b,j}}$$

and then by (34) we have that (33) holds for α_j, β_j, γ_j and η_j which satisfy $\alpha_j, \gamma_j > 0$. We still require $\beta_j < D_j$ and $4\alpha_j(D_j - \beta_j) > 1$. Make the following transformation

$$\xi_j = \frac{P_{j,22}}{4\tau_{b,j}}, \quad \sigma_j = \frac{\xi_j}{\gamma_j}, \quad z_j = \frac{\tau_{g,j}\gamma_j}{P_{j,11}} \quad (35)$$

so that

$$\xi_j > 0, \qquad 0 < \sigma_j < 1, \qquad 0 < z_j < 1. \quad (36)$$

Hence the conditions $\beta_j < D_j$ and $4\alpha_j(D_j - \beta_j) > 1$ become

$$D_j - \frac{L_j^2\xi_j}{4\sigma_jz_j(1 - z_j)} > 0 \quad (37)$$

$$16\xi_j(1 - \sigma_j)\left(D_j - \frac{L_j^2\xi_j}{4\sigma_jz_j(1 - z_j)}\right) > 1. \quad (38)$$

Subject to (36)(37), the maximum of the left-hand-side of (38) is $\frac{D_j^2}{L_j^2}$, attained at $z_j = \frac{1}{2}$, $\frac{L_j^2\xi_j}{\sigma_j} = \frac{D_j}{2}$, and $\sigma_j = \frac{1}{2}$. By Condition 4 we have $\frac{D_j^2}{L_j^2} > 1$, i.e., there exists a (ξ_j, σ_j, z_j) that satisfies (36)–(38). Through inverse transformation of (35), we can find positive definite, diagonal matrices P_j for all $j \in \mathcal{G}$ such that (33) holds, which finishes the proof. ■

Theorem 2 provides a sufficient condition for stability. Its proof depends on the particular (diagonal) structure of P_j and the particular form (33) for the bound of the time derivative of $\mathcal{V}_j$. Hence the condition in Theorem 2 may be conservative, and it is possible that a closed-loop equilibrium may still be stable even if Conditions 3 and 4 are not satisfied.

V. CASE STUDY

We illustrate the performance of the proposed control through a simulation of the IEEE New England test system shown in Fig. 1.

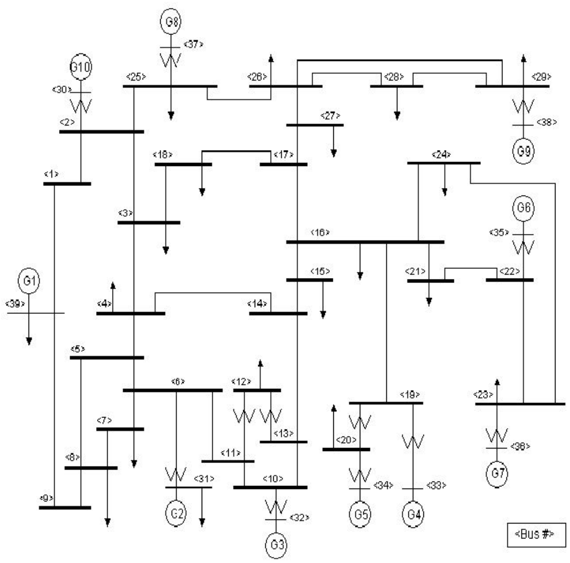

Fig. 1. IEEE New England test system [39].

This system has 10 generators and 39 buses, and a total load of about 60 per unit (pu) where 1 pu represents 100 MVA. Details about this system including parameter values can be found in Power System Toolbox [39], which we use to run the simulation in this section. Compared to the model (2)–(4), the simulation model is more detailed and realistic, with transient generator dynamics, excitation and flux decay dynamics, changes in voltage and reactive power over time, and lossy transmission lines, et cetera.

The primary frequency control of generator or load j is designed with cost function $c_j(p_j) = \frac{R_j}{2}(p_j - p_j^{\text{set}})^2$, where p_j^{set} is the power injection at the setpoint, an initial equilibrium point solved from static power flow problem. By choosing this cost function, we try to minimize the deviations of power injections from the setpoint, and have the control $p_j = \left[p_j^{\text{set}} - \frac{1}{R_j}\omega_j\right]_{\underline{p}_j}^{\overline{p}_j}$ from (15)(16) [3]. We consider the following two cases in which the generators and loads have different control capabilities and hence different $[\underline{p}_j, \overline{p}_j]$:

1) All the 10 generators have $[\underline{p}_j, \overline{p}_j] = [p_j^{\text{set}}(1-c), p_j^{\text{set}}(1+c)]$, and all the loads are uncontrollable;
2) Generators 2, 4, 6, 8, 10 (which happen to provide half of the total generation) have the same bounds as in case (1). Generators 1, 3, 5, 7, 9 are uncontrollable, and all the loads have $[\underline{p}_j, \overline{p}_j] = [p_j^{\text{set}}(1+c/2), p_j^{\text{set}}(1-c/2)]$, if we suppose $p_j^{\text{set}} \leq 0$ for loads $j \in \mathcal{L}$.

Hence cases (1) and (2) have the same total control capacity across the network. Case (1) only has generator control while in case (2) the set of generators and the set of loads each has half of the total control capacity. We select $c = 10\%$, which implies the total control capacity is about 6 pu. For all $j \in \mathcal{N}$, the feedback gain $1/R_j$ is selected as $25p_j^{\text{set}}$, which is a typical value in practice meaning a frequency change of 0.04 pu (2.4 Hz) causes the change of power injection from zero all the way to the setpoint. Note that this control is the same as frequency droop control, which implies that indeed frequency droop control implicitly solves an OFC problem with quadratic cost functions we use here. However, our controller design is more flexible by allowing a larger set of cost functions.

In the simulation, the system is initially at the setpoint with 60 Hz frequency. At time $t = 0.5$ second, buses 4, 15, 16 each makes 1 pu step change in their real power consumptions, causing the frequency to drop. Fig. 2 shows the frequencies of all the 10 generators under the two cases above, (1) with red and (2) with black. We see in both cases that frequencies of different generators have relatively small differences during transient, and are synchronized towards the new steady-state frequency. Compared with generator-only control, the combined generator-and-load control improves both the transient and steady-state frequency, even though the total control capacities in both cases are the same.

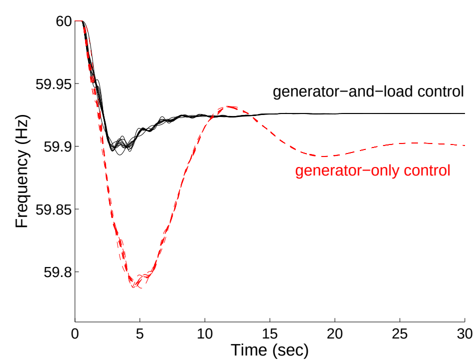

Fig. 2. Frequencies of all the 10 generators under case (1) only generators are controlled (red) and case (2) both generators and loads are controlled (black). The total control capacities are the same in these two cases.

VI. CONCLUSION AND FUTURE WORK

We have presented a systematic method to jointly design generator and load-side primary frequency control, by formulating an optimal frequency control (OFC) problem to characterize the desired equilibrium points of the closed-loop system. OFC minimizes the total generation cost and user disutility subject to power balance over entire network. The proposed control is completely decentralized, depending only on local frequency. Stability analysis for the closed-loop system with Lyapunov method has led to a sufficient condition for any equilibrium point to be asymptotically stable. A simulation shows that the combined generator-and-load control improves both transient and steady-state frequency, compared to the traditional control on generators only, even when the total control capacity remains the same.

[3]Only the load control p_j for $j \in \mathcal{L}$ is written since the generator control p_j^c for $j \in \mathcal{G}$ takes the same form.

We have got the stability condition of any closed-loop equilibrium without characterizing its attraction region, and particularly the change of attraction region due to closing the loop. It is an interesting topic for the future. It would also be useful to understand how conservative the sufficient stability condition in Theorem 2 is and derive less conservative conditions. Moreover, much work remains to extend these results to more detailed dynamical models like those in [22], [39], [40] with flux decay dynamics, time-varying voltage magnitudes, reactive power flows, and lossy transmission lines. Finally, we are interested in studying the performance of the proposed primary frequency control when it operates jointly with current schemes of secondary and tertiary control.

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