Optimal Charging of Plug-in Hybrid Electric Vehicles in Smart Grids

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Abstract— **Plug-in hybrid electric vehicles (PHEVs) play an important role in making a greener future. Given a group of PHEVs distributed across a power network equipped with the smart grid technology (e.g. wireless communication devices), the objective of this paper is to study how to schedule the charging of the PHEV batteries. To this end, we assume that each battery must be fully charged by a pre-specified time, and that the charging rate can be time-varying at discrete-time instants. The scheduling problem for the PHEV charging can be augmented into the optimal power flow (OPF) problem to obtain a joint OPF-charging (dynamic) optimization. A solution to this highly nonconvex problem optimizes the network performance by minimizing the generation and charging costs while satisfying the network, physical and inelastic-load constraints. A global optimum to the joint OPF-charging optimization can be found efficiently in polynomial time by solving its convex dual problem whenever the duality gap is zero for the joint OPF-charging problem. It is shown in a recent work that the duality gap is expected to be zero for the classical OPF problem. We build on this result and prove that the duality gap is zero for the joint OPF-charging optimization if it is zero for the classical OPF problem. The results of this work are applied to the IEEE 14 bus system.**

I. Introduction

Plug-in hybrid electric vehicles (PHEVs) are becoming more popular as we move toward a greener future. PHEVs are equipped with rechargeable batteries, which can be charged by connecting it to an electric power source. Several studies have shown that PHEVs produce less CO2 and other pollutants over their entire fuel cycle, compared to both conventional and hybrid electric vehicles [1], [2], [3]. The charging of PHEVs, however, is a challenging problem due to two reasons: (i) simultaneous charging of several PHEVs located in the same area can overload the network, and (ii) PHEVs should ideally be charged during off-peak hours when the power delivery cost is at its lowest.

Upgrading conventional grids to smart grids has been the center of attention in the past few years. The design of a smart grid needs new fundamental theories as well as modern technologies [4], [5], [6], [7], [8]. A building block of a smart grid is a communication network, which allows for real-time data exchange between consumers and power providers. In particular, some (price-elastic) residential loads at each house, such as washing machines and PHEVs, should be equipped with wireless communication devices to be able to communicate with the smart meter of the house, which is in contact with the utility company.

Assume that each PHEV battery is plugged into a smart outlet, whose output power is controllable. The question of interest is, at what time-varying rate should each PHEV battery be charged to minimize the charging cost? In other words, having known the necessary information about the capacities of the PHEVs batteries and the number of them in each area, what is the optimal schedule for the charging of all batteries over some given period?

Power network can be connected to two types of loads:

- *Price-inelastic load:* The exact power requested by this type of load must be provided. This corresponds to the standard loads in a conventional grid
- *Price-elastic load:* The power delivered to this type of load depends on the current price. This corresponds to the PHEV batteries in a smart grid.

With these two types of loads in the power network model, the objective is to minimize a weighted sum of the total generation cost and the PHEV charging cost while satisfying the network and physical constraints. To formulate this problem, one needs to build on the optimal power flow (OPF) problem.

The OPF problem aims to find a steady-state operating point of a power system that minimizes a cost function such as the generation cost or power loss [9], [10], [11], [12], [13]. The OPF problem is highly nonconvex, and therefore many optimization techniques have been studied in the past 5 decades to find a near-optimal or local solution of this problem. However, due to the non-convexity of the OPF problem, these algorithms suffer from several drawbacks including robustness issues, inability to find a global optimum, and scalability. Interestingly, the recent work [14] shows that the OPF problem associated with a practical power system is highly structured so that a global solution to this problem is likely to be attainable in polynomial time via solving a convex problem. Indeed, that work suggests solving the dual of the OPF problem—a semidefinite program (SDP)—and then recovering a global solution to the OPF problem from a solution to its dual. This is possible provided the duality gap is zero. The work [14] derives a zero-duality-gap condition for the OPF problem, which is satisfied by the IEEE systems with 14, 30, 57, 118 and 300 buses, and is expected to hold widely in practice for other systems.

In this work, we augment the optimal PHEV charging problem into the OPF problem and introduce a joint OPF-charging (dynamic) optimization. This problem consists of a number of OPF problems coupled with each other in time. It also include elastic loads. The joint OPF-charging optimization is inherently harder than a classical (static) OPF problem. In this work, we propose to solve the dual of the joint OPF-charging optimization, which is convex and can be solved efficiently in polynomial time. However, this technique can find a solution to the original joint OPF-charging problem only if the duality gap is zero. The main contribution of this paper is to show that if the topology of the power network is such

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that the duality gap is zero for the classical OPF problem, then it is still zero for the joint OPF-charging problem. It is worth mentioning that we consider PHEV batteries as loads here, but our results can be easily generalized to the case when these batteries are allowed to inject power into the grid if need be.

The paper is organized as follows. The problem is formulated in section II and the main results are given in section III. The results of this work are simulated on the IEEE test system with 14 buses in section IV. Finally, some concluding remarks are provided in section V.

Notations: We use the following notations throughout the paper:

- $\mathbf{i}$: The imaginary unit.
- T: The transpose operator.
- Re$\{.\}$ and Im$\{.\}$: The operators returning the real and imaginary parts of a complex matrix, respectively.
- $\mathcal{R}$: The set of real numbers.
- $|\cdot|$: The absolute value operator.

II. Problem formulation

Consider a power network with the set of buses $\mathcal{N} := \{1, 2, ..., n\}$, the set of generator buses $\mathcal{G} \subseteq \mathcal{N}$ and the set of flow lines $\mathcal{L} \subseteq \mathcal{N} \times \mathcal{N}$. Suppose that each bus $k \in \mathcal{N}$ is connected to a price-inelastic conventional load as well as a price-elastic PHEV battery. With no loss of generality, assume that each PHEV battery can absorb only active power and with an adjustable rate (this can be realized via a smart outlet). The goal is to optimize the operation of the power grid over a time interval $[1, T]$, for some natural number T, such that all PHEV batteries are charged at the end of the horizon T. To this end, assume that the controllable parameters in the power grid, such as the active power and voltage magnitude at a generator bus or the charging rate for a PHEV battery, can be adjusted only at the discrete time instants $1, 2, ..., T-1$. Assume also that the price-inelastic loads are known and fixed over each time interval $[t, t+1]$ for every $t \in \{1, ..., T-1\}$. Note that although each bus is connected to only one PHEV battery in this work, the results can be easily generalized to the case when a number of PHEV batteries are co-located at the same bus. We introduce some notations in the following for every $t \in \{1, ..., T\}$:

- $P_{I_k}[t]$ and $Q_{I_k}[t]$: The given active and reactive parts of the price-inelastic load connected to bus $k \in \mathcal{N}$ at time t (these numbers are zero whenever such a load are absent on bus k).
- $P_{E_k}[t]$ and $Q_{E_k}[t]$: The control variables corresponding to active and reactive parts of the price-elastic load connected to bus $k \in \mathcal{N}$ at time t (these numbers are zero whenever bus k is not connected to any PHEV battery).
- $P_{G_k}[t]$ and $Q_{G_k}[t]$: The control variables corresponding to active and reactive parts of the power generated at bus $k \in \mathcal{G}$ at time t.
- $V_k[t]$: Complex voltage at bus $k \in \mathcal{N}$ at time t.
- $S_{lm}[t]$: The apparent power transferred at time t from bus $l \in \mathcal{N}$ to the rest of the network through line $(l, m) \in \mathcal{L}$.
- $f_k(P_{G_k}[t]) = c_{k2} P_{G_k}[t]^2 + c_{k1} P_{G_k}[t] + c_{k0}$: A quadratic cost function with known nonnegative coefficients accounting for the cost of active power generation at bus $k \in \mathcal{G}$ at time t.

Given $t \in \{1, ..., T-1\}$, define the sets

$$\mathbf{V}[t] = \big(V_k[t]:\ \forall k \in \mathcal{N}\big), \quad \mathbf{P}_E[t] = \big(P_{E_k}[t]:\ \forall k \in \mathcal{N}\big)$$
$$\mathbf{P}_G[t] = \big(P_{G_k}[t]:\ \forall k \in \mathcal{G}\big), \quad \mathbf{Q}_G[t] = \big(Q_{G_k}[t]:\ \forall k \in \mathcal{G}\big)$$

We study the joint OPF-charging problem: Minimize

$$\sum_{t=1}^{T-1} \sum_{k \in \mathcal{G}} f_k(P_{G_k}[t]) + \sum_{t=1}^{T-1} \sum_{k \in \mathcal{N}} \alpha_k[t] P_{E_k}[t] \tag{1}$$

over $\mathbf{V}[t], \mathbf{P}_G[t], \mathbf{Q}_G[t], \mathbf{P}_E[t]$, $\forall t \in \{1, ..., T-1\}$, subject to the power balance equations at all buses, the physical constraints

$$P_k^{\min} \leq P_{G_k}[t] \leq P_k^{\max}, \quad \forall k \in \mathcal{G} \tag{2a}$$
$$Q_k^{\min} \leq Q_{G_k}[t] \leq Q_k^{\max}, \quad \forall k \in \mathcal{G} \tag{2b}$$
$$V_k^{\min} \leq |V_k[t]| \leq V_k^{\max}, \quad \forall k \in \mathcal{N} \tag{2c}$$
$$|S_{kl}[t]| \leq S_{kl}^{\max} \quad \forall (k, l) \in \mathcal{L} \tag{2d}$$

for every $t \in \{1, ..., T-1\}$ and the PHEV charging constraints

$$\sum_{t=1}^{T-1} P_{E_k}[t] = C_k, \quad \forall k \in \mathcal{G} \tag{3a}$$
$$P_{E_k}[t] \geq 0, \quad \forall k \in \mathcal{G},\ t \in \{1, ..., T-1\} \tag{3b}$$

where

- $\alpha_k[t]$ is a known scalar accounting for the linear charging cost of the PHEV battery connected to bus k at time t.
- $P_k^{\min}, P_k^{\max}, Q_k^{\min}, Q_k^{\max}, V_k^{\min}, V_k^{\max}, S_{kl}^{\max}$ are some given physical limits on the parameters of the power system.
- C_k is the capacity of the PHEV battery connected to bus k.

The objective of this paper is to show that if the duality gap is zero for the classical OPF problem, then a global solution to the above joint OPF-charging problem can be found efficiently in polynomial time by solving a convex optimization problem. Note that a detailed background on the notion of *zero duality gap for the OPF problem* can be found in the recent works [14] and [15]. Note also that the results of this paper can be easily generalized to more complicated joint OPF-charging problems with additional constraints (e.g. ramp-up and ramp-down constraints on the power generations).

III. Main results

By denoting the standard basis vectors in $\mathcal{R}^n$ as $e_1, e_2, ..., e_n$ and the admittance matrix of the equivalent circuit model of the power network as Y, we define a number of matrices for every $k \in \mathcal{N}$, $(l,m) \in \mathcal{L}$ and $t \in \{1,...,T-1\}$:

$$\begin{aligned}
Y_k &:= e_k e_k^T Y \\
Y_{lm} &:= (\bar{y}_{lm} + y_{lm}) e_l e_l^T - (y_{lm}) e_l e_m^T \\
\mathbf{Y}_k &:= \frac{1}{2} \begin{bmatrix} \mathrm{Re}\{Y_k + Y_k^T\} & \mathrm{Im}\{Y_k^T - Y_k\} \\ \mathrm{Im}\{Y_k - Y_k^T\} & \mathrm{Re}\{Y_k + Y_k^T\} \end{bmatrix} \\
\mathbf{Y}_{lm} &:= \frac{1}{2} \begin{bmatrix} \mathrm{Re}\{Y_{lm} + Y_{lm}^T\} & \mathrm{Im}\{Y_{lm}^T - Y_{lm}\} \\ \mathrm{Im}\{Y_{lm} - Y_{lm}^T\} & \mathrm{Re}\{Y_{lm} + Y_{lm}^T\} \end{bmatrix} \\
\bar{\mathbf{Y}}_k &:= \frac{-1}{2} \begin{bmatrix} \mathrm{Im}\{Y_k + Y_k^T\} & \mathrm{Re}\{Y_k - Y_k^T\} \\ \mathrm{Re}\{Y_k^T - Y_k\} & \mathrm{Im}\{Y_k + Y_k^T\} \end{bmatrix} \\
\bar{\mathbf{Y}}_{lm} &:= \frac{-1}{2} \begin{bmatrix} \mathrm{Im}\{Y_{lm} + Y_{lm}^T\} & \mathrm{Re}\{Y_{lm} - Y_{lm}^T\} \\ \mathrm{Re}\{Y_{lm}^T - Y_{lm}\} & \mathrm{Im}\{Y_{lm} + Y_{lm}^T\} \end{bmatrix} \\
M_k &:= \begin{bmatrix} e_k e_k^T & 0 \\ 0 & e_k e_k^T \end{bmatrix} \\
\mathbf{X}[t] &:= \begin{bmatrix} \mathrm{Re}\{\mathbf{V}[t]\}^T & \mathrm{Im}\{\mathbf{V}[t]\}^T \end{bmatrix}^T
\end{aligned}$$

where $\bar{y}_{lm}$ denotes the shunt element at bus l associated with the Π model of the line (l,m) and y_{lm} denotes the series element in this Π model. Given $t \in \{1,...,T-1\}$, let $P_{k,\text{inj}}[t]$ and $Q_{k,\text{inj}}[t]$ denote the net active and reactive powers injected to bus $k \in \mathcal{N}$ at time t, i.e.,

$$P_{k,inj}[t] := \begin{cases} P_{G_k}[t] - P_{I_k}[t] - P_{E_k}[t] & \text{if} \quad k \in \mathcal{G} \\ -P_{I_k}[t] - P_{E_k}[t] & \text{if} \quad k \in \mathcal{N} \backslash \mathcal{G} \end{cases}$$

$$Q_{k,inj}[t] := \begin{cases} Q_{G_k}[t] - Q_{I_k}[t] & \text{if} \quad k \in \mathcal{G} \\ -Q_{I_k}[t] & \text{if} \quad k \in \mathcal{N} \backslash \mathcal{G} \end{cases}$$

It can be verified that the following relations hold for every $k \in \mathcal{N}$, $(l,m) \in \mathcal{L}$ and $t \in \{1,...,T-1\}$ [14]:

$$\begin{aligned}
P_{k,inj}[t] &= \mathrm{trace}\{\mathbf{Y}_k \mathbf{X}[t]\mathbf{X}[t]^T\} \\
Q_{k,inj}[t] &= \mathrm{trace}\{\bar{\mathbf{Y}}_k \mathbf{X}[t]\mathbf{X}[t]^T\} \\
|V_k[t]|^2 &= \mathrm{trace}\{M_k \mathbf{X}[t]\mathbf{X}[t]^T\} \\
|S_{lm}[t]|^2 &= \left(\mathrm{trace}\{\mathbf{Y}_{lm} \mathbf{X}[t]\mathbf{X}[t]^T\}\right)^2 \\
&\quad + \left(\mathrm{trace}\{\bar{\mathbf{Y}}_{lm} \mathbf{X}[t]\mathbf{X}[t]^T\}\right)^2
\end{aligned} \tag{4}$$

Using (4), the joint OPF-charging problem formalized in (1), (2) and (3) can be formulated in terms of the complex bus voltages and the charging rates at times $1,...,T-1$. This leads to the following optimization problem.

Joint OPF-charging optimization: Minimize

$$\begin{aligned}
&\sum_{t=1}^{T-1} \sum_{k \in \mathcal{G}} f_k\left(\mathrm{trace}\{\mathbf{Y}_k \mathbf{X}[t]\mathbf{X}[t]^T\} + P_{I_k}[t] + P_{E_k}[t]\right) \\
&+ \sum_{t=1}^{T-1} \sum_{k \in \mathcal{N}} \alpha_k[t] P_{E_k}[t]
\end{aligned} \tag{5}$$

over the variables $\mathbf{X}[1],...,\mathbf{X}[T-1] \in \mathcal{R}^{2n}$ and $P_{E_k}[1],...,P_{E_k}[T-1] \in \mathcal{R}$, $\forall k \in \mathcal{N}$, subject to the following constraints for every $k \in \mathcal{N}$, $(l,m) \in \mathcal{L}$ and $t \in \{1,...,T-1\}$:

$$\begin{aligned}
P_k^{\min} - P_{I_k}[t] - P_{E_k}[t] &\leq \mathrm{trace}\{\mathbf{Y}_k \mathbf{X}[t]\mathbf{X}[t]^T\} \\
&\leq P_k^{\max} - P_{I_k}[t] - P_{E_k}[t]
\end{aligned} \tag{6a}$$

$$Q_k^{\min} - Q_{I_k}[t] \leq \mathrm{trace}\{\bar{\mathbf{Y}}_k \mathbf{X}[t]\mathbf{X}[t]^T\} \leq Q_k^{\max} - Q_{I_k}[t] \tag{6b}$$

$$\left(V_k^{\min}\right)^2 \leq \mathrm{trace}\{M_k \mathbf{X}[t]\mathbf{X}[t]^T\} \leq \left(V_k^{\max}\right)^2 \tag{6c}$$

$$\begin{aligned}
&\mathrm{trace}\{\mathbf{Y}_{lm} \mathbf{X}[t]\mathbf{X}[t]^T\}^2 \\
&\qquad + \mathrm{trace}\{\bar{\mathbf{Y}}_{kl} \mathbf{X}[t]\mathbf{X}[t]^T\}^2 \leq \left(S_{lm}^{\max}\right)^2
\end{aligned} \tag{6d}$$

$$\sum_{\tau=1}^{T-1} P_{E_k}[\tau] = C_k \tag{6e}$$

$$P_{E_k}[t] \geq 0 \tag{6f}$$

where $P_k^{\min}, P_k^{\max}, Q_k^{\min}, Q_k^{\max}$ are defined as zero if $k \in \mathcal{N} \backslash \mathcal{G}$.

The joint OPF-charging problem is nonconvex in the sense that its feasibility region can be nonconvex and/or disconnected. However, its dual can be cast as a semidefinite programming (SDP) optimization and subsequently solved efficiently in polynomial time. Then, a globally optimal solution to the joint OPF-charging problem can be retrieved from a dual solution if the duality gap is zero for the joint OPF-charging problem (see [14] for a similar argument). The main result of this paper, presented in the next theorem, is that the duality gap is zero for the joint OPF-charging problem if the duality gap is zero for the topology Y with respect to the classical OPF problem. To this end, let a precise definition of the zero duality gap with respect to the topology Y be presented first.

Definition 1: Define the classical OPF problem as the minimization of $\sum_{k \in \mathcal{G}} f_k(P_{G_k})$ over $\mathbf{V}, \mathbf{P}_G, \mathbf{Q}_G$ subject to the constraints that every bus $k \in \mathcal{N}$ delivers some given complex power $P_{D_k} + Q_{D_k}\mathrm{i}$ to its load and that

$$\begin{aligned}
P_k^{\min} &\leq P_{G_k} \leq P_k^{\max}, \quad \forall k \in \mathcal{G} \\
Q_k^{\min} &\leq Q_{G_k} \leq Q_k^{\max}, \quad \forall k \in \mathcal{G} \\
V_k^{\min} &\leq |V_k| \leq V_k^{\max}, \quad \forall k \in \mathcal{N} \\
|S_{lm}| &\leq S_{lm}^{\max}, \quad \forall (l,m) \in \mathcal{L}
\end{aligned}$$

(note that the classical OPF problem has no price-elastic load and time evolution $t = 1,...,T-1$).

Definition 2: It is said that *the duality gap is zero for the topology Y with respect to the classical OPF problem* if the duality gap is zero for the classical OPF problem, independent of the values of the loads and physical limits [15].

Theorem 1: Suppose that the duality gap is zero for the topology Y with respect to the classical OPF problem. Then, the duality gap is zero for the joint OPF-charging problem so that a global optimum of this optimization problem can be found from a solution to its convex dual problem.

Proof: The idea developed in [15] will be adopted here to prove this theorem. Consider the problem of minimizing

$$\begin{aligned}
&\sum_{t=1}^{T-1} \sum_{k \in \mathcal{G}} f_k\left(\mathrm{trace}\{\mathbf{Y}_k W[t]\} + P_{I_k}[t] + P_{E_k}[t]\right) \\
&+ \sum_{t=1}^{T-1} \sum_{k \in \mathcal{N}} \alpha_k[t] P_{E_k}[t]
\end{aligned} \tag{7}$$

over the positive semidefinite matrices $W[1], ..., W[T-1] \in \mathcal{R}^{2n\times 2n}$ and the scalars $P_{E_k}[1], ..., P_{E_k}[T-1] \in \mathcal{R}$, $\forall k \in \mathcal{N}$, subject to the following constraints for every $k \in \mathcal{N}$, $(l,m) \in \mathcal{L}$ and $t \in \{1, ..., T-1\}$:

$$P_k^{\min} - P_{I_k}[t] - P_{E_k}[t] \le \text{trace}\big\{\mathbf{Y}_k W[t]\big\} \tag{8a}$$

$$\text{trace}\big\{\mathbf{Y}_k W[t]\big\} \le P_k^{\max} - P_{I_k}[t] - P_{E_k}[t] \tag{8b}$$

$$Q_k^{\min} - Q_{I_k}[t] \le \text{trace}\big\{\bar{\mathbf{Y}}_k W[t]\big\} \tag{8c}$$

$$\text{trace}\big\{\bar{\mathbf{Y}}_k W[t]\big\} \le Q_k^{\max} - Q_{I_k}[t] \tag{8d}$$

$$\big(V_k^{\min}\big)^2 \le \text{trace}\big\{M_k W[t]\big\} \le \big(V_k^{\max}\big)^2 \tag{8e}$$

$$\text{trace}\big\{\mathbf{Y}_{lm} W[t]\big\}^2 + \text{trace}\big\{\bar{\mathbf{Y}}_{kl} W[t]\big\}^2 \le \big(S_{lm}^{\max}\big)^2 \tag{8f}$$

$$\sum_{\tau=1}^{T-1} P_{E_k}[\tau] = C_k \tag{8g}$$

$$P_{E_k}[t] \ge 0 \tag{8h}$$

It can be verified that this optimization problem is related to the joint OPF-charging problem in two ways:

- The above optimization can be obtained from the joint OPF-charging problem by changing the variable $\mathbf{X}[t]\mathbf{X}[t]^T$ as $W[t]$, $\forall t \in \{1, ..., T-1\}$, and imposing a positive semi-definiteness constraint on $W[t]$.
- The above optimization is the dual of the dual of the joint OPF-charging problem.

It follows from the above properties that the duality gap is zero for the joint OPF-charging problem if the optimization problem given in (7) and (8) has a solution $(W^{\text{opt}}[1], W^{\text{opt}}[1], \cdots, W^{opt}[T-1])$ such that $W^{\text{opt}}[t]$ has rank 1 for every $t \in \{1, ..., T-1\}$. To prove the existence of such a solution, let $(W^{\text{opt}}[1], \cdots, W^{opt}[T-1])$ and $(\mathbf{P}_E^{\text{opt}}[1], \cdots, \mathbf{P}_E^{\text{opt}}[T-1])$ be an arbitrary minimizer of the latter optimization problem. Now, consider the problem of minimizing

$$\sum_{k\in\mathcal{G}} f_k\bigg(\text{trace}\big\{\mathbf{Y}_k W[1]\big\} + P_{I_k}[1] + P_{E_k}^{\text{opt}}[1]\bigg) \tag{9}$$

over $W[1]$ subject to the following constraints for every $k \in \mathcal{G}$ and $(l,m) \in \mathcal{L}$:

$$\begin{aligned}
&\text{trace}\big\{\mathbf{Y}_k W[1]\big\} = \text{trace}\big\{\mathbf{Y}_k W^{opt}[1]\big\} \\
&\text{trace}\big\{\bar{\mathbf{Y}}_k W[1]\big\} = \text{trace}\big\{\bar{\mathbf{Y}}_k W^{opt}[1]\big\} \\
&\text{trace}\big\{M_k W[1]\big\} = \text{trace}\{M_k W^{opt}[1]\} \\
&\text{trace}\big\{\mathbf{Y}_{lm} W[1]\big\}^2 + \text{trace}\big\{\bar{\mathbf{Y}}_{lm} W[1]\big\}^2 \le (S_{lm}^{\max})^2
\end{aligned} \tag{10}$$

The matrix $W^{\text{opt}}[1]$ is a solution to the above optimization problem because $(W^{\text{opt}}[1], \cdots, W^{opt}[T-1])$ is a minimizer of the optimization problem given in (7) and (8). On the other hand, the above optimization problem is the dual of the dual of a classical OPF problem at time $t=1$, which minimizes

$$\sum_{k\in\mathcal{G}} f_k(P_{G_k}[1])$$

over $\mathbf{V}[0], \mathbf{P}_G[1], \mathbf{Q}_G[1]$ subject to the constraints that every bus $k \in \mathcal{N}$ delivers the power $P_{I_k}[1] + P_{E_k}^{\text{opt}}[1] + Q_{I_k}[1]\text{i}$ to its load and that

$$\begin{aligned}
&P_{G_k} = \text{trace}\big\{\mathbf{Y}_k W^{opt}[1]\big\} + P_{I_k}[1] + P_{E_k}^{\text{opt}}[1], \quad \forall k \in \mathcal{G} \\
&Q_{G_k} = \text{trace}\big\{\bar{\mathbf{Y}}_k W^{opt}[1]\big\} + Q_{I_k}[1], \quad \forall k \in \mathcal{G} \\
&|V_k| = \sqrt{\text{trace}\{M_k W^{opt}[1]\}}, \quad \forall k \in \mathcal{N} \\
&|S_{lm}| \le S_{lm}^{\max}, \quad \forall (l,m) \in \mathcal{L}
\end{aligned}$$

Hence, since the duality gap is assumed to be zero for the classical OPF problem, the dual of the dual of the above optimization problem, i.e. the optimization given in (9) and (10), has a rank-one solution. This simply implies that the minimizer $W^{\text{opt}}[1]$ of this optimization problem can be assumed to be rank-one. The same argument can be continued for the remaining matrices $W^{\text{opt}}[2], ..., W^{\text{opt}}[T-1]$ to conclude that they are all rank-one. This completes the proof. ■

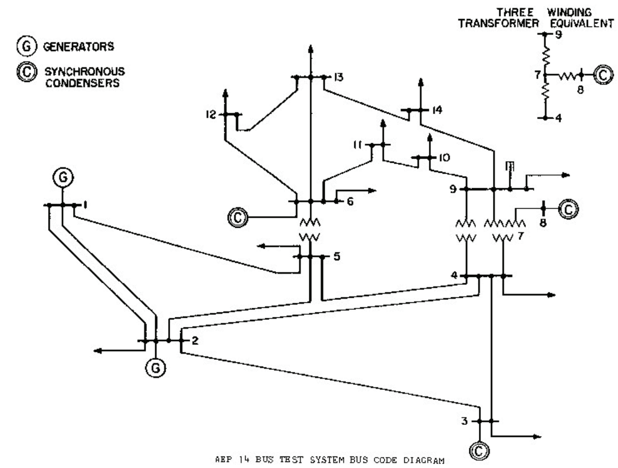

Fig. 1. IEEE 14-bus system studied in Section IV [17].

IV. Simulation Results

Consider the IEEE 14-bus system depicted in Figure 1, where the circuit specifications and the physical limits are given in the library of the toolbox MATPOWER [16]. This system has 5 generators connected to buses $1,2,3,6,8$. Assume that each of the non-generator buses $4,5,7,9,10,11,12,13,14$ is connected to a PEHV with the capacity of 5 per unit. Enumerate the batteries of these vehicles as $1,2,...,9$. Assume that all batteries are plugged in at time $t=1$ and must be fully charged by time $t=11$, where the charging rate of each battery can be varied (controlled) only at the discrete time instants $1,2,...,10$.

Aside from the elastic PEHV loads, suppose that each bus $k \in \{1,2,...,14\}$ is connected to an inelastic load as well, which increases at the discrete times $1,2,...,10$ based on the relation

$$P_{I_k}[t] = \left(1 + \frac{t}{30}\right) \times P_k, \quad t = 1,2,...,10 \tag{11}$$

where $(P_1, ..., P_{14})$ is equal to the load profile given in the library of the toolbox MATPOWER for the IEEE 14-bus system. The goal is to optimize the controllable parameters of the power network, such as the active power supplied by a generator or the charging rate of a battery, which can be

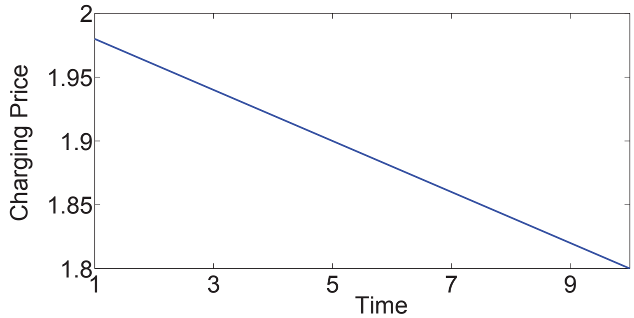

Fig. 2. The time-varying pricing vector used in Section IV.

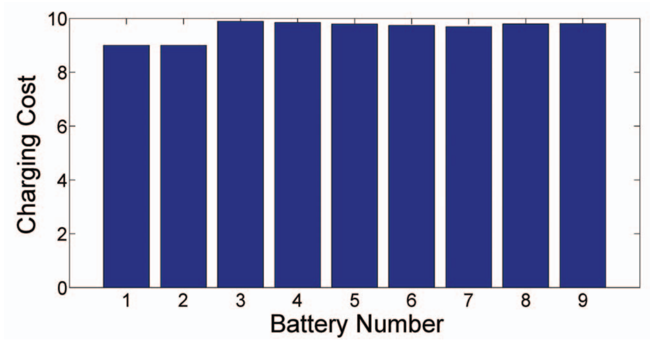

Fig. 3. Charging cost in the time-varying pricing case.

modified only at the time instants $1, 2, ..., 10$. To this end, we aim to minimize the cost function

$$\sum_{t=1}^{10} \sum_{k \in \mathcal{G}} P_{G_k}[t] + \sum_{t=1}^{10} \sum_{k \in \mathcal{N} \setminus \mathcal{G}} \alpha[t] P_{E_k}[t] \tag{12}$$

for different values of the pricing vector $(\alpha[1], ..., \alpha[10])$. Regarding this cost function, one can note that:

- The generation cost is considered to be the total active power generated by all the generators over the time horizon $[1, 11]$.
- The pricing vector of each battery is assumed to be independent of its bus number.

Two scenarios will be considered in the sequel.

Time-invariant pricing: In this case, assume that $\alpha[1], ..., \alpha[10]$ are all identical and equal to 2. Consider the dual of the joint OPF-charging problem, which is a convex optimization problem and can be solved efficiently in polynomial time. It can be observed that the duality gap is zero in this case and a global solution to the joint OPF-charging problem can be found from the dual solution. This leads to the optimal charging rates depicted in Figure 4. These plots demonstrate that since the charging price is time-independent and the inelastic loads increase over time, the optimal strategy is to fully charge all the batteries by the time $t = 7$.

Time-varying pricing: In this case, the numbers $\alpha_1, ..., \alpha_{10}$ form an increasing sequence with the values

$$\alpha_t = 2 - \frac{t}{50}, \quad t = 1, 2, ..., 10 \tag{13}$$

as plotted in Figure 2. This pricing vector incentivizes the batteries not to be charged very quickly and introduces a non-uniform charging schedule. Indeed, it can be shown that the duality gap is still zero in this case and a globally optimal schedule is to charge the batteries based on the plots given in Figure 5. The charging cost corresponding to each battery is given in Figure 3. Notice that even though the pricing mechanism is slowly time-varying, it incentivized batteries 1 and 2 to wait and get charged only at the final time slot. It is interesting to note that none of the batteries are charged from time 7 to 10.

V. Conclusions

This paper deals with the plug-in hybrid electric vehicles (PHEVs) in a power network, which can be regarded as price-elastic loads. To optimally schedule the charging of the PHEVs, a dynamic OPF problem can be solved, which is associated with both elastic and inelastic loads. This problem, referred to as *joint OPF-charging problem*, can be solved efficiently in polynomial time (through its dual) if the duality gap is zero for this problem. In this work, we prove that the duality gap is zero for the joint OPF-charging problem whenever it is zero for the classical (static) OPF problem with no elastic loads. Since recent work suggests that the duality gap for the classical OPF problem is likely to be zero, the result of this paper opens up an efficient way to globally optimize a fundamental optimization problem for smart grids.

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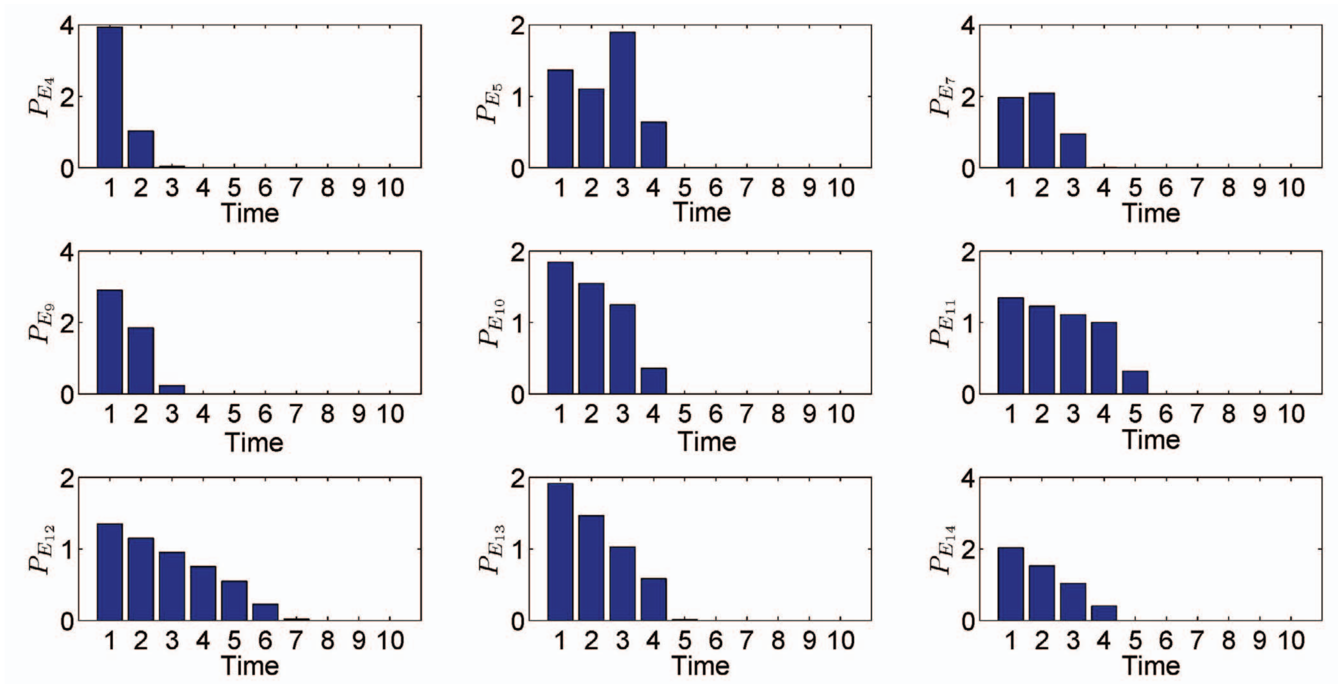

Fig. 4. Optimal charging rates for the PEHVs in the time-invariant pricing case.

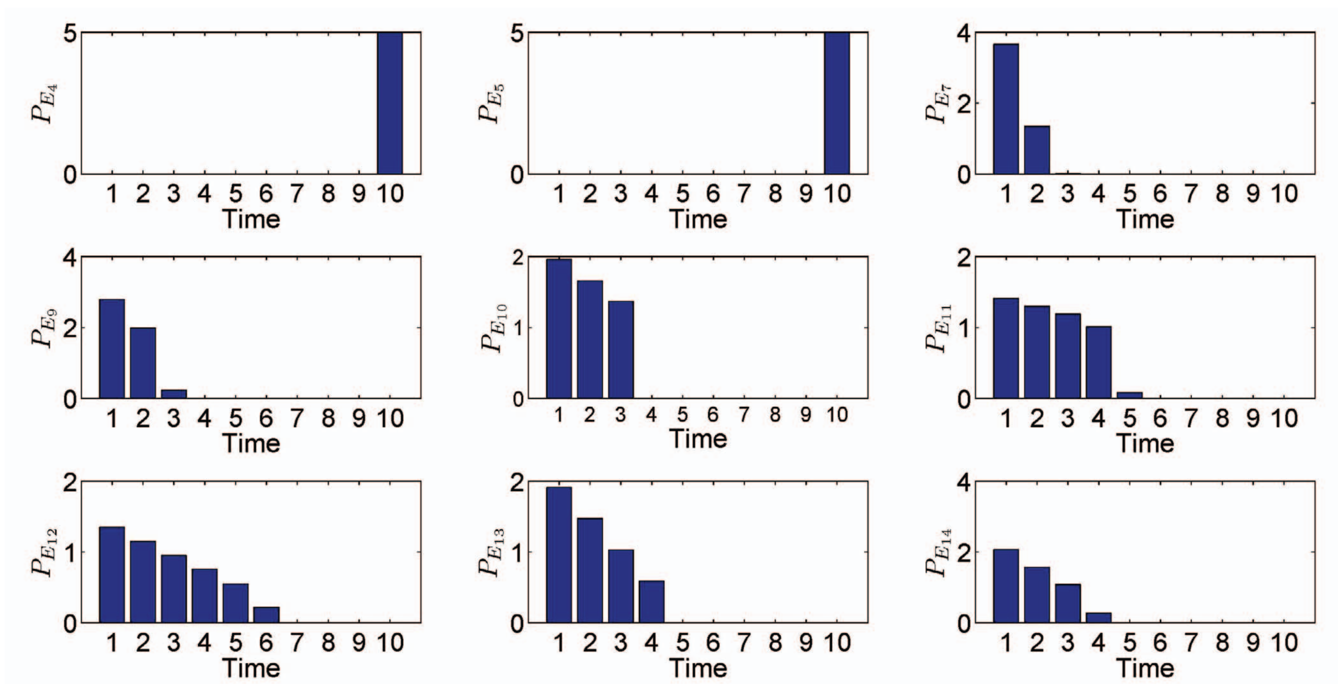

Fig. 5. Optimal charging rates for the PEHVs in the time-varying pricing case.

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