Power flow optimization using positive quadratic programming ⋆

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Abstract: The problem to minimize power losses in an electrical network subject to voltage and power constraints is in general hard to solve. However, it has recently been discovered that semidefinite programming relaxations in many cases enable exact computation of the global optimum. Here we point out a fundamental reason for the successful relaxations, namely that the passive network components give rise to matrices with nonnegative offdiagonal entries. Recent progress on quadratic programming with Metzler matrix structure can therefore be applied.

Keywords: Power flow, optimization, relaxation analysis

1. INTRODUCTION

The optimal power flow (OPF) problem aims to find an optimal operating point of a power system, that minimizes an appropriate cost function such as generation cost or transmission loss subject to certain constraints on power and voltage variables (Momoh, 2001). Since the pioneering work (Carpentier, 1962), the OPF problem has been extensively studied in the literature and numerous algorithms have been proposed for solving this highly nonlinear problem (Huneault and Galiana, 1991; Torres and Quintana, 2000; H. Wang and Thomas, 2007). Approaches include linear programming, Newton Raphson, quadratic programming, nonlinear programming, Lagrange relaxation, interior point methods, artificial intelligence, artificial neural network, fuzzy logic, genetic algorithms, evolutionary programming and particle swarm optimization (Momoh, 2001; El-Hawary and Adapa, 1999a,b; Pandya and Joshi, 2008). A good number of these methods are based on the Karush-Kuhn-Tucker (KKT) necessary conditions, which due to the nonconvex problem formulations can only guarantee a locally optimal solution (H. Wei and Yokoyama, 1998). This nonconvexity is partially due to the cross products of voltage variables corresponding to disparate buses. In the past decade, much attention has been paid to devising efficient algorithms with guaranteed performance for the OPF problem. For instance, the recent papers (W. M. Lin and Zhan, 2008) and (Q. Y. Jiang and Cao, 2009) propose nonlinear interior-point algorithms for an equivalent current injection model of the problem. An improved implementation of the automatic differentiation technique for the OPF problem is studied in the recent work (Q. Jiang and Cao, 2010). In an effort to convexify the OPF problem, it is shown in (Jabr, 2006) that the load flow problem of a radial distribution system can be modeled as a convex optimization problem in the form of a conic program. Nonetheless, the results fail to hold for a meshed network, due to the presence of arctangent equality constraints (Jabr, 2008). Nonconvexity appears in more sophisticated power problems such as the stability constrained OPF problem where the stability at the operating point is an extra constraint (D. Gan and Zimmerman, 2000; H. R. Cai and Wong, 2008) or the dynamic OPF problem where the dynamics of the generators are also taken into account (Xie and Song, 2002; Xia and Chan, 2006).

In (Lavaei and Low, 2010), it was proposed to solve the Lagrangian dual of the OPF problem and recover the desired solution from a dual optimum. This approach was applied successfully to several examples and possible explanations were discussed. In this paper, we point out the connection to a class of quadratic programming problems with nonconvex quadratic constraints for which semidefinite relaxations are always exact (Kim and Kojima, 2003). A rigorous mathematical statement is given and its application to DC power networks is explained in the next section. AC networks are not covered by Theorem 1, but the semidefinite relaxations have still turned out to be exact for the IEEE benchmark systems treated in section 5. Concluding remarks are given in section 6.

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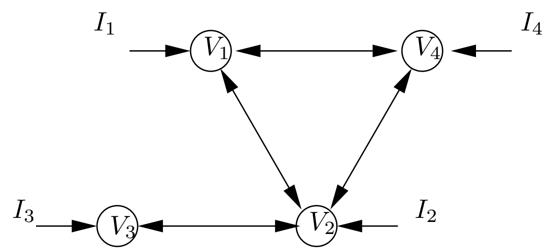

Fig. 1. A power distribution network

Notations: We introduce the following notations:

- i : The imaginary unit.
- **R**: The set of real numbers.
- $\mathrm{Re}\{\cdot\}$ and $\mathrm{Im}\{\cdot\}$: The operators returning the real and imaginary parts of a complex matrix.
- $*$: The conjugate transpose operator.
- T : The transpose operator.
- $\succeq$ and $\preceq$: The matrix inequality signs in the positive semidefinite sense (i.e. given two symmetric matrices A and B, $A \succeq B$ implies $A - B$ is a positive semidefinite matrix, meaning that its eigenvalues are all nonnegative).
- Trace: The matrix trace operator.
- $|\cdot|$: The absolute value operator.
- For any vector x, x_i generally denotes the ith component.

2. OPTIMAL POWER FLOW IN DC NETWORKS

Consider a DC power transmission network as in Figure 1. All nodes are subject to constraints of the form $I_k V_k \leq P_k$. For generating nodes P_k represents the generator capacity. For power consuming loads I_k and P_k are negative and $-P_k$ represents the power demand.

Every connection has a known admittance $y_{jk} = y_{kj} \geq 0$. In particular, the current flowing from node 1 to node 2 equals $y_{12}(V_1 - V_2)$. Writing Kirchhoff's current law for all nodes in Figure 1 gives

$$\underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix}}_{I} = \underbrace{\begin{bmatrix} y_{12}+y_{14} & -y_{12} & 0 & -y_{14} \\ -y_{21} & y_{21}+y_{23}+y_{24} & -y_{23} & -y_{24} \\ 0 & -y_{32} & y_{32} & 0 \\ -y_{41} & -y_{42} & 0 & y_{41}+y_{42} \end{bmatrix}}_{Y} \underbrace{\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}}_{V} \tag{1}$$

Suppose every link has a has a capacity bound L_{ij} on the transferred power and every node has upper and lower bounds on the voltage according to $V_k^{\min} \leq V_k \leq V_k^{\max}$. Then the problem to minimize the power losses in the network subject to constaints on power demands, voltage and link capacities can be written

$$\begin{array}{ll} \text{Minimize} & I_1 V_1 + \cdots + I_N V_N \\ \text{subject to} & I = YV \text{ with } V_k I_k \leq P_k \\ & V_k^{\min} \leq V_k \leq V_k^{\max} \\ & y_{jk}(V_k - V_j)^2 \leq L_{jk} \\ & \text{for } j, k = 1, \ldots, N \end{array}$$

This is a quadratic optimization problem with quadratic constraints. The constraints are not convex in the variables $V_1, \ldots, V_N$, so the problem could look intractable at first. However, a closer look reveals that both the objective and the constraints are concave in $(V_1^2, \ldots, V_N^2)$ (Megretski, 2010). This is because every product $V_j V_k$ is the geometric mean of two such variables, hence concave. The fact that all $y_{jk} \geq 0$ is essential.

Another way to get a convex formulation of the OPF problem is by convex relaxation. The following result from (Kim and Kojima, 2003, Theorem 3.1) shows that if a nonconvex quadratic programming problem is defined by *Metzler matrices* (matrices with nonnegative off-diagonal elements), then it can be solved exactly using a semidefinite programming relaxation.

Proposition 1. (Positive Quadratic Programming). Let $M_0, \ldots, M_K \in \mathbf{R}^{n \times n}$ be Metzler and $b_1, \ldots, b_K \in \mathbf{R}$. Then

$$\begin{array}{llcll} \max & x^T M_0 x & = & \max & \mathrm{trace}(M_0 X) \\ \text{s.t.} & x \in \mathbf{R}_+^n & & \text{s.t.} & X \succeq 0 \\ & x^T M_k x \geq b_k & & & \mathrm{trace}(M_k X) \geq b_k \\ & k = 1, \ldots, K & & & k = 1, \ldots, K \end{array} \tag{2}$$

Proof. Every x satisfying the constraints on the left hand side of (2) corresponds to a matrix $X = xx^T$ satisfying the constraints on the right hand side. This shows that the right hand side of (2) is at least as big as the left.

On the other hand, let $X = (x_{ij})$ be any positive definite matrix. In particular, the diagonal elements $x_{11}, \ldots, x_{nn}$ are non-negative and $x_{ij} \leq \sqrt{x_{ii} x_{jj}}$. Let $x = (\sqrt{x_{11}}, \ldots, \sqrt{x_{nn}})$. Then the matrix xx^T has the same diagonal elements as X, but has off-diagonal elements $\sqrt{x_{ii} x_{jj}}$ instead of x_{ij}. The fact that xx^T has off-diagonal elements at least as big as those of X, together with the assumption that the matrices M_k are Metzler, gives

$$x^T M_k x \geq \mathrm{trace}(M_k X) \qquad k = 1, \ldots, K$$

This shows that the left hand side of (2) is at least as big as the right and the proof is complete. □

We believe that the convex reformulations of the OPF problem for DC networks presented above are of significant practical importance. In addition to real DC transmission networks, the results are relevant for analysis of power markets where DC networks are used as approximations of AC networks. For example, the Lagrange multiplier corresponding to the constraint $I_k V_k \leq P_k$ can be interpreted as the optimal price of power at node k.

3. OPTIMAL POWER FLOW IN AC NETWORKS

Consider an AC power network with n buses, labeled $1, ..., n$, where all buses are possibly directly connected to loads, but only the first m buses are directly connected to generators. For $k \in \{1, ..., n\}$ and $l \in \{1, ..., m\}$, define the following quantities:

- P_k^d and Q_k^d (real-valued): Active and reactive loads at buses k, respectively. They are given fixed demands.
- P_l^g and Q_l^g (real-valued): Active and reactive powers generated at buses l, respectively. They are optimization variables.
- V_k (complex-valued): Voltages at buses k. They are optimization variables.
- $f_l(P_l^g) = c_{l2}(P_l^g)^2 + c_{l1} P_l^g + c_{l0}$ (real-valued): Cost functions associated with generators l, where c_{l2}, c_{l1}, c_{l0} are nonnegative numbers.

Derive the circuit model of the power network by replacing every transmission line and transformer with their equivalent Π models (Momoh, 2001). In this circuit model, let y_{kl} be the mutual admittance between buses k and l, and y_{kk} be the admittance-to-ground at bus k, for every $l, k \in \{1, ..., n\}$. Denote the admittance matrix of this equivalent circuit model with Y, which is an $n \times n$ complex-valued matrix whose (l, k) entry is equal to $-y_{lk}$ if $l \neq k$ and $y_{ll} + \sum_{p \in \mathcal{N}(l)} y_{lp}$ otherwise, where $\mathcal{N}(l)$ is the set of buses that are directly connected to bus l. Denote by the column vector $V := (V_k,\ \ k = 1, \ldots, n)$ the complex voltages. Define the current vector $I := YV = (I_k,\ \ k = 1, ..., n)$. Let $P^g := (P_l^g,\ \ l = 1, \ldots, m)$ and $Q^g := (Q_l^g,\ \ l = 1, \ldots, m)$.

The classical optimal power flow (OPF) problem is:

OPF:

$$\min_{V, P^g, Q^g} \sum_{l=1}^{m} f_l(P_l^g) \tag{3}$$

subject to

$$P_l^{\min} \le P_l^g \le P_l^{\max}, \quad l = 1, 2, ..., m \tag{4a}$$

$$Q_l^{\min} \le Q_l^g \le Q_l^{\max}, \quad l = 1, 2, ..., m \tag{4b}$$

$$V_k^{\min} \le |V_k| \le V_k^{\max}, \quad k = 1, 2, ..., n \tag{4c}$$

$$V_l I_l^* = (P_l^g - P_l^d) + (Q_l^g - Q_l^d)\mathrm{i}, \quad l = 1, 2, ..., m \tag{4d}$$

$$V_k I_k^* = -P_k^d - Q_k^d \mathrm{i}, \quad k = m+1, ..., n \tag{4e}$$

The inequalities (4a), (4b) and (4c) limit the power and voltage variables to within the given bounds $P_l^{\min}$, $P_l^{\max}$, $Q_l^{\min}$, $Q_l^{\max}$, $V_k^{\min}$, $V_k^{\max}$, whereas the last two equations (4d) and (4e) express the physical constraints imposed by the network.

Though not stated explicitly in the results that follow, we assume the following condition to hold throughout the paper:

C0: (i) OPF (3)–(4) is feasible. Moreover, $V = 0$ is not a feasible point of OPF.
(ii) The admittance matrix Y is symmetric ($y_{ij} = y_{ji}$) and has two important properties: the off-diagonal entries of the matrix $\mathrm{Re}\{Y\}$ are all nonpositive, and the off-diagonal entries of the matrix $\mathrm{Im}\{Y\}$ are all nonnegative.

Assumption C0(i) is to avoid triviality. Assumption C0(ii) always holds in standard power systems where the resistance, capacitance and inductance in the Π model of transmission lines are positive.

4. ALGORITHM

The voltage constraints (4c) and the network constraints (4d)–(4e) are the sources of nonconvexity that makes OPF generally hard. Our approach is to consider a convex relaxation of the problem, which can be solved efficiently. To state our main result, we need the following notations.

Eliminating the variables $P_l^g = \mathrm{Re}\{Y_l I_l^*\} + P_l^d$ and $Q_l^g = \mathrm{Im}\{Y_l I_l^*\} + Q_l^d$ using the network constraints (4d) and (4e), we can write the OPF problem in terms only of the complex voltages V (noting $I = YV$). Extend the definition of $P_k^{\min}, P_k^{\max}, Q_k^{\min}, Q_k^{\max}$ to $k \in \{m+1, ..., n\}$, with $P_k^{\min} = P_k^{\max} = Q_k^{\min} = Q_k^{\max} = 0$ if $k \in \{m + 1, ..., n\}$. Let $e_1, e_2, ..., e_n$ denote the standard basis vectors in $\mathbf{R}^n$. For every $k = 1, 2, ..., n$, define $M_k \in \mathbf{R}^{2n \times 2n}$ as a diagonal matrix whose entries are all equal to zero, except for its (k, k) and $(n+k, n+k)$ entries that are equal to 1. Define also

$$Y_k := e_k e_k^* Y$$

$$\mathbf{Y}_k := \frac{1}{2} \begin{bmatrix} \mathrm{Re}\{Y_k + Y_k^T\} & \mathrm{Im}\{Y_k^T - Y_k\} \\ \mathrm{Im}\{Y_k - Y_k^T\} & \mathrm{Re}\{Y_k + Y_k^T\} \end{bmatrix}$$

$$\bar{\mathbf{Y}}_k := \frac{-1}{2} \begin{bmatrix} \mathrm{Im}\{Y_k + Y_k^T\} & \mathrm{Re}\{Y_k - Y_k^T\} \\ \mathrm{Re}\{Y_k^T - Y_k\} & \mathrm{Im}\{Y_k + Y_k^T\} \end{bmatrix}$$

Define the variables for the dual problem as a $6n$-dimensional real vector:

$$x := (\lambda_k^{\min}, \lambda_k^{\max}, \bar{\lambda}_k^{\min}, \bar{\lambda}_k^{\max}, \mu_k^{\min}, \mu_k^{\max},\ k = 1, ..., n)$$

and a $2m$-dimensional real vector

$$r := (r_{l1}, r_{l2},\ l = 1, ..., m)$$

Define the affine function

$$h(x, r) := \sum_{k=1}^{n} \Big\{ \lambda_k^{\min} P_k^{\min} - \lambda_k^{\max} P_k^{\max} + \boldsymbol{\lambda}_k P_k^d + \bar{\lambda}_k^{\min} Q_k^{\min} - \bar{\lambda}_k^{\max} Q_k^{\max} + \bar{\boldsymbol{\lambda}}_k Q_k^d + \mu_k^{\min} \left(V_k^{\min}\right)^2 - \mu_k^{\max} \left(V_k^{\max}\right)^2 \Big\} + \sum_{l=1}^{m} (c_{l0} - r_{l2})$$

where the bold variables are defined in terms of (x, r) as: for $k = 1, ..., n$

$$\boldsymbol{\lambda}_k := \begin{cases} -\lambda_k^{\min} + \lambda_k^{\max} + c_{k1} + 2\sqrt{c_{k2}}\, r_{k1} & \text{if } k = 1, ..., m \\ -\lambda_k^{\min} + \lambda_k^{\max} & \text{otherwise} \end{cases}$$

$$\bar{\boldsymbol{\lambda}}_k := -\bar{\lambda}_k^{\min} + \bar{\lambda}_k^{\max}$$

$$\boldsymbol{\mu}_k := -\mu_k^{\min} + \mu_k^{\max}$$

Instead of the nonconvex OPF problem, we propose solving the following convex problem.

Dual OPF:

$$\max_{x \ge 0, r} \ h(x, r) \tag{5}$$

subject to

$$\sum_{k=1}^{n} \left(\boldsymbol{\lambda}_k \mathbf{Y}_k + \bar{\boldsymbol{\lambda}}_k \bar{\mathbf{Y}}_k + \boldsymbol{\mu}_k M_k \right) \succeq 0 \tag{6a}$$

$$\begin{bmatrix} 1 & r_{l1} \\ r_{l1} & r_{l2} \end{bmatrix} \succeq 0, \quad l = 1, 2, ..., m \tag{6b}$$

This semidefinite program is the dual of an equivalent form of OPF. See (Lavaei and Low, 2011). It is therefore convex and can be solved efficiently. This motivates the following approach to solving OPF.

Algorithm for Solving OPF:

(1) Compute a solution $(x^{\text{opt}}, r^{\text{opf}})$ of Dual OPF (5)–(6).
(2) If the optimal value of Dual OPF is $+\infty$, then OPF is infeasible.
(3) Compute any nonzero vector $\begin{bmatrix} U_1^T & U_2^T \end{bmatrix}^T$ in the null space of the $2n \times 2n$ positive semidefinite matrix

$$A^{\text{opt}} := \sum_{k=1}^{n} \left(\boldsymbol{\lambda}_k^{\text{opt}} \mathbf{Y}_k + \bar{\boldsymbol{\lambda}}_k^{\text{opt}} \bar{\mathbf{Y}}_k + \boldsymbol{\mu}_k^{\text{opt}} M_k \right) \tag{7}$$

(4) Compute an optimal solution V^{opt} of OPF as

$$V^{\text{opt}} = (\zeta_1 + \zeta_2 \mathrm{i})(U_1 + U_2 \mathrm{i}) \tag{8}$$

by solving for ζ_1 and ζ_2 from optimality conditions.

(5) Verify that $V^{\rm opt}$ satisfies all the constraints of OPF (3)–(4) and that the resulting objective value of OPF equals the optimal value of Dual OPF (zero duality gap).

We make several remarks. First, provided OPF is feasible, the null space of $A^{\rm opt}$ has an even dimension of at least 2. Hence Step 3 of the Algorithm will always yield a nonzero vector $\begin{bmatrix} U_1^T & U_2^T \end{bmatrix}^T$. Second, having found U_1 and U_2, the scalars ζ_1 and ζ_2 can be identified from the first order optimality (KKT) condition for Dual OPF or the feasibility condition for OPF. For instance, the voltage angle at the swing bus being zero introduces an equation in terms of ζ_1 and ζ_2. If, in addition, $(\mu_k^{\min})^{\rm opt}$ (respectively, $(\mu_k^{\max})^{\rm opt}$) turns out to be nonzero for some $k \in \{1, 2, ...n\}$, then the relation $|V_k^{\rm opt}| = V_k^{\min}$ (respectively, $|V_k^{\rm opt}| = V_k^{\max}$) must hold by complementary slackness, which provides another equation relating ζ_1 to ζ_2. Third, the weak duality theorem implies that the optimal value of OPF is greater than or equal to that of its dual. Hence, Step 2 detects when OPF is infeasible. Even when OPF is feasible, there is generally a nonzero duality gap and an optimal solution to OPF may not be recoverable from an optimal dual solution. However, if $V^{\rm opt}$ computed in Step 4 indeed is primal feasible as verified in Step 5, then duality gap is zero and $V^{\rm opt}$ is indeed optimal for OPF. This is the case with all the IEEE benchmark examples described in Section 5, and hence all of them can be solved efficiently by the above Algorithm.

Indeed, the following *sufficient* condition guarantees that the Algorithm finds an optimal solution of OPF:

C1: There exists a dual optimal solution $(x^{\rm opt}, r^{\rm opt})$ such that the $2n \times 2n$ positive semidefinite matrix $A^{\rm opt}$ in (7) has a zero eigenvalue of multiplicity 2.

In this case, the null space of $A^{\rm opt}$ has dimension 2.

If condition C1 holds, then

(1) There is no duality gap between OPF and Dual OPF.
(2) Given any vector $\begin{bmatrix} U_1^T & U_2^T \end{bmatrix}^T$ in the null space of $A^{\rm opt}$, the voltages $V^{\rm opt}$ calculated in (8) is indeed optimal for OPF.

See (Lavaei and Low, 2011) for details.

5. POWER SYSTEM EXAMPLES

This section illustrates our results through two examples. Example 1 uses the IEEE benchmark systems archived at (University of Washington) to show the practicality of our result. Since the systems analyzed in Example 1 are so large that the specific values of the optimal solution cannot be provided in the paper, some smaller examples are analyzed in Example 2 with more details.

There are two main findings from this exercise. First, the duality gap is zero for all the systems we have tried, even when the sufficient condition C1 is not satisfied. We verify this by following the Algorithm in Section 4 to solve Dual OPF and compute the voltages. In all cases, the voltages obtained are feasible for Optimization 1 and achieve a primal objective value that is equal to the optimal objective value of Optimization 2. By weak duality theorem, the duality gap is zero and the voltages are optimal for OPF. Second, condition C1 is essentially satisfied: when it is violated, the violation is due to the simplifying modeling assumption that transformers have zero resistance. If a small resistance (10^{-5} per unit) is added to each of these transformers, condition C1 is satisfied for all IEEE benchmark systems.

The results of this section are attained using the following software tools:

- The MATLAB-based toolbox "YALMIP" (together with the solver "SEDUMI") is used to solve the dual of the OPF problem (Optimization 2), which is in the form of a linear-matrix-inequality optimization problem (Löfberg, 2004).
- The software toolbox "MATPOWER" is used to solve the OPF problem in Example 1 for the sake of comparison. The data for the IEEE benchmark systems analyzed in this example is extracted from the library of this toolbox (R. D. Zimmerman and Thomas, 2009).
- The software toolbox "PSAT" is used to draw and analyze the power networks given in Example 2 (Milano, 2005).

5.1 Example 1: IEEE benchmark systems

We have solved all IEEE systems with 14, 30, 57, 118 and 300 buses using the method developed in this paper, where the goal is to minimize either the total generation cost or the power loss. However, due to space restrictions, the details will be provided here only for two cases: (i) the loss minimization for the IEEE 30-bus system, and (ii) the total generation cost minimization for the IEEE 118-bus system.

IEEE 30-bus system First, consider the OPF problem for the IEEE 30-bus system, where the objective is to minimize the total power generated by the generators. When the original Optimization 2 is solved, the four smallest eigenvalues of the matrix

$$A^{\rm opt} = \begin{bmatrix} H_1(\mathbf{\Lambda}^{\rm opt}, \bar{\mathbf{\Lambda}}^{\rm opt}, \mathbf{\Gamma}^{\rm opt}) & H_2(\mathbf{\Lambda}^{\rm opt}, \bar{\mathbf{\Lambda}}^{\rm opt}, \mathbf{\Gamma}^{\rm opt}) \\ -H_2(\mathbf{\Lambda}^{\rm opt}, \bar{\mathbf{\Lambda}}^{\rm opt}, \mathbf{\Gamma}^{\rm opt}) & H_1(\mathbf{\Lambda}^{\rm opt}, \bar{\mathbf{\Lambda}}^{\rm opt}, \mathbf{\Gamma}^{\rm opt}) \end{bmatrix}$$

would be obtained as $0, 0, 0, 0$. Since the number of zero eigenvalues is 4, condition C1 is violated. To understand the underlying reason, one should note that the network is composed of three regions connected to each other via some transformers. This implies that if each line of the circuit is replaced by its resistive part, the resulting resistive graph will not be connected (since the lines with transformers are assumed to have no resistive parts). Thus, the graph induced by $\text{Re}\{Y\}$ is not strongly connected. This is an issue with all the IEEE benchmark systems. This can be easily fixed by adding a little resistance to each transformer, say on the order of 10^{-5} (per unit). After this modification to the real part of Y, the four smallest eigenvalues of the matrix $A^{\rm opt}$ turn out to be $0, 0, 0.0075, 0.0075$; i.e. the zero eigenvalues resulting from the non-connectivity of the resistive graph have disappeared. Condition C1 is satisfied and the corresponding vector of optimal voltages can be recovered.

Note that, for $k = 1, \ldots, n$,

$$\boldsymbol{\lambda}_k \in [1, 1.0426], \quad \bar{\boldsymbol{\lambda}}_k \in [0, 0.0152], \quad \boldsymbol{\mu}_k \in [0, 0.0098],$$

Hence

- $\boldsymbol{\lambda}_k$'s are all positive and around 1.
- $\bar{\boldsymbol{\lambda}}_k$'s are all positive and around 0.
- $\boldsymbol{\mu}_k$'s are all very close to 0.

Moreover, the maximum absolute values of the entries of $H_2(\boldsymbol{\Lambda}^{\text{opt}}, \bar{\boldsymbol{\Lambda}}^{\text{opt}}, \boldsymbol{\Gamma}^{\text{opt}})$ is 0.0867, whereas the average absolute values of the nonzero entries of $H_1(\boldsymbol{\Lambda}^{\text{opt}}, \bar{\boldsymbol{\Lambda}}^{\text{opt}}, \boldsymbol{\Gamma}^{\text{opt}})$ is 4.1201.

IEEE 118-bus system Consider now the problem of minimizing the total generation cost for the IEEE 118-bus system. After adding some small resistance to certain entries of Re$\{Y\}$ to make the induced graph strongly connected, the four smallest eigenvalues of the matrix

$$\begin{bmatrix} H_1(\boldsymbol{\Lambda}^{\text{opt}}, \bar{\boldsymbol{\Lambda}}^{\text{opt}}, \boldsymbol{\Gamma}^{\text{opt}}) & H_2(\boldsymbol{\Lambda}^{\text{opt}}, \bar{\boldsymbol{\Lambda}}^{\text{opt}}, \boldsymbol{\Gamma}^{\text{opt}}) \\ -H_2(\boldsymbol{\Lambda}^{\text{opt}}, \bar{\boldsymbol{\Lambda}}^{\text{opt}}, \boldsymbol{\Gamma}^{\text{opt}}) & H_1(\boldsymbol{\Lambda}^{\text{opt}}, \bar{\boldsymbol{\Lambda}}^{\text{opt}}, \boldsymbol{\Gamma}^{\text{opt}}) \end{bmatrix}$$

are 0, 0, 1.3552, 1.3552. Hence, condition C1 is satisfied and OPF can be solved by solving Dual OPF. The optimal variables normalized by $c_{l1} = 40$ satisfy, for $k = 1, \ldots, n$,

$$\frac{\boldsymbol{\lambda}_k}{c_{l1}} \in [0.8858, 1.0356], \quad \frac{\bar{\boldsymbol{\lambda}}_k}{c_{l1}} \in [-0.0063, 0.0118], \quad \frac{\boldsymbol{\mu}_k}{c_{l1}} \in [0, 0.1894]$$

As before, $(\frac{\boldsymbol{\lambda}_k}{c_{l1}}, \frac{\bar{\boldsymbol{\lambda}}_k}{c_{l1}}, \frac{\boldsymbol{\mu}_k}{c_{l1}})$ are around (1, 0, 0). In addition, $\boldsymbol{\lambda}_k$'s are all positive and most of $\bar{\boldsymbol{\lambda}}_k$ are positive (more than 100 of them). As the last property, the maximum of the absolute values of the entries of $H_2(\boldsymbol{\Lambda}^{\text{opt}}, \bar{\boldsymbol{\Lambda}}^{\text{opt}}, \boldsymbol{\Gamma}^{\text{opt}})$ is 13.8613, whereas the average of the absolute values of the nonzero entries of $H_1(\boldsymbol{\Lambda}^{\text{opt}}, \bar{\boldsymbol{\Lambda}}^{\text{opt}}, \boldsymbol{\Gamma}^{\text{opt}})$ is 237.3938. Thus, H_2 is negligible compared to H_1 as before.

The computation on the IEEE benchmark examples were all finished in a few seconds and the number of iterations for each example was between 5 and 20. Note that although Optimization 2 is convex and there is no convergence problem regardless of what initial point is used, the number of iterations needed to converge mainly depends on the choice of starting point. It is worth mentioning that when different algorithms implemented in Matpower were applied to these systems, some of the constraints are violated at the optimal point probably due to the large-scale and non-convex nature of the OPF problem. However, no constraint violation have occurred by solving the dual of the OPF problem due to its convexity.

5.2 Example 2: small systems

The IEEE test systems in the previous example operate in a normal condition when the optimal bus voltages are close to each other both in magnitude and phase. This example illustrates that condition C1 is satisfied even in the absence of such a normal operation. Consider three distributed power systems, referred to as Systems 1, 2 and 3. Systems 2 and 3 are radial, while System 1 has a loop. The detailed specifications of these systems are provided in Table 1 in per unit for the voltage rating 400kV and the power rating 100MVA, in which $\bar{z}_{ij}$ and $\bar{y}_{ij}$ denote the series impedance and the shunt capacitance of the Π model of the transmission line connecting buses $i, j \in \{1, 2, 3, 4\}$.

Table 1. Parameters of systems in Example 2.

Parameters	System 1	System 2	System 3
$\bar{z}_{12}$	$0.05 + 0.25$i	$0.1 + 0.5$i	$0.10 + 0.1$i
$\bar{z}_{13}$	$0.04 + 0.40$i	None	None
$\bar{z}_{23}$	$0.02 + 0.10$i	$0.02 + 0.20$i	$0.01 + 0.1$i
$\bar{z}_{14}$	None	None	$0.01 + 0.2$i
$\bar{y}_{12}$	0.06i	0.02i	0.06i
$\bar{y}_{13}$	0.05i	None	None
$\bar{y}_{23}$	0.02i	0.02i	0.02i
$\bar{y}_{14}$	None	None	0.02i

Table 2. Constraints for systems in Example 2.

Constraints	System 1	System 2	System 3
$P_2^d + Q_2^d$i	$0.95 + 0.4$i	$0.7 + 0.02$i	$0.9 + 0.02$i
$P_3^d + Q_3^d$i	$0.9 + 0.6$i	$0.65 + 0.02$i	$0.6 + 0.02$i
$P_4^d + Q_4^d$i	None	None	$0.9 + 0.02$i
$V_1^{\max}$	1.05	1.4	1

Table 3. OPF parameters from Optimization 2.

Recovered Parameters	System 1	System 2	System 3
V_1	$1.05\angle 0°$	$1.4\angle 0°$	$1\angle 0°$
V_2	$0.71\angle{-20.11°}$	$1.10\angle{-25.73°}$	$0.78\angle{-10.58°}$
V_3	$0.68\angle{-21.94°}$	$1.08\angle{-31.96°}$	$0.76\angle{-16.31°}$
V_4	None	None	$0.95\angle{-10.82°}$
P_{loss}	0.2193	0.1588	0.3877
Q_{loss}	1.2944	0.7744	0.5343

Table 4. Lagrange multipliers obtained by solving Optimization 2 for systems in Example 2.

Lagrange Multipliers	System 1	System 2	System 3
$\boldsymbol{\lambda}_2$	1.3809	1.4028	1.7176
$\boldsymbol{\lambda}_3$	1.4155	1.4917	1.7900
$\boldsymbol{\lambda}_4$	None	None	1.0207
$\bar{\boldsymbol{\lambda}}_2$	0.4391	0.2508	0.1764
$\bar{\boldsymbol{\lambda}}_3$	0.4955	0.2633	0.1858
$\bar{\boldsymbol{\lambda}}_4$	None	None	0.0061
$\boldsymbol{\mu}_1$	0.0005	0.0001	0.0005

The goal is to minimize the active power injected at slack bus 1 while satisfying the constraints given in Table 2.

Optimization 2 is solved for each of these systems, and it is observed that condition C1 always holds. The optimal solution of OPF recovered from the solution of Optimization 2 are provided in Table 3 (P_{loss} and Q_{loss} in the table represent the total active and reactive power losses, respectively). It is interesting to note that although different buses have very disparate voltage magnitudes and phases, the duality gap is still zero. The optimal solution of Optimization 2 is summarized in Table 4 to demonstrate that the Lagrange multipliers corresponding to active and reactive power constraints are positive.

As another scenario, let the desired voltage magnitude at the slack bus of System 1 be changed from 1.05 to 1. It can be verified that the optimal value of Optimization 2 becomes $+\infty$, which simply implies that the corresponding OPF problem is infeasible.

We repeated several hundred times this example by randomly choosing the parameters of the systems over a wide range of values. In all these trials, the Algorithm prescribed in Section 3 always found a globally optimal solution of the OPF problem or detected its infeasibility.

6. CONCLUSIONS

We have studied the classical optimal power flow (OPF) problem that is notorious for its difficult nonlinear constraints. For DC networks we have proven that the problem has a convex semi-definite programming relaxation which is always equivalent to the original problem. For AC networks, a similar semi-definite relaxation yields the exact solution for the IEEE benchmark systems with 14, 30, 57, 118 and 300 buses, after a small resistance (10^{-5} per unit) is added to every transformer.

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